

Last lecture we ended with the problem of showing the following:

$$\neg(\neg((A \cup B) \cap C) \cup \neg B) = (B \cap C)$$

One way is to use a membership table:

A	B	C	$B \cap C$	$\neg((A \cup B) \cap C)$	$\neg B$	$\neg(\neg((A \cup B) \cap C) \cup \neg B)$
0	0	0	0	1	1	0
0	0	1	0	1	1	0
0	1	0	0	1	0	0
0	1	1	1	0	0	1
1	0	0	0	1	1	0
1	0	1	0	0	1	0
1	1	0	0	1	0	0
1	1	1	1	0	0	1

Next, we could use Set Laws to show the two expressions to be equivalent:

$$\begin{aligned}
 & \neg(\neg((A \cup B) \cap C) \cup \neg B) \\
 = & \neg\neg((A \cup B) \cap C) \cap \neg\neg B && \text{(De Morgan's)} \\
 = & ((A \cup B) \cap C) \cap B && \text{(Double Negation)} \\
 = & ((A \cup B) \cap B) \cap C && \text{(Associate \& Commutative)} \\
 = & B \cap C && \text{(Absorption)}
 \end{aligned}$$

For this particular problem, we could show that each set is a subset of the other. However, for this particular problem, that is rather tedious. I will give other examples of this nature to illustrate this proof method later in the lecture.

## Example of a Set Proof

If  $A \cap B = \emptyset$ , then  $B \subseteq \neg A$ .

The standard method we will use with any if-then statement is to prove it directly, so to speak. We know that the statement is automatically true if the “if” clause is false. In fact, to establish the validity of the statement, we must only show that whenever the “if” clause is true, then the “then” clause must ALSO be true.

In order to do this, we can do the following:

- 1) Assume that the if clause is true.
- 2) Under this assumption, make deductions using the laws and rules you know
- 3) Hopefully show that the “then” clause follows logically from the deductions in part 2.

So here we are going to assume that  $A \cap B = \emptyset$ .

Under this assumption we want to show that  $B \subseteq \neg A$  is always true.

If we have that  $A \cap B = \emptyset$ , then we know that they do NOT share any common elements. Symbolically we have the following:

$$\neg(\exists x \mid x \in A \wedge x \in B)$$

Now, under this assumption, we need to show that  $B \subseteq \neg A$ . In order to show that one set is a subset of another, we must show that any element in the first set belongs in the second. In essence, for an arbitrary element  $x$ , we must show the following:

if  $x \in B$  then  $x \in \neg A$ .

To show this, we simply assume that  $x \in B$ . But, from before, we already KNOW that  $\neg(\exists x \mid x \in A \wedge x \in B)$ . What this means in English is that there is NO element that belongs to both A and B. If we know that the particular element  $x$  belongs in B, then it CAN NOT belong in A. Symbolically we have  $x \notin A$ .

But, we have for all elements that  $x \in A \vee x \in \neg A$ . Since  $x \in A$  is false from what we deduced above, we MUST HAVE  $x \in \neg A$ , which is exactly what we needed to prove.

So, whenever  $A \cap B = \emptyset$ , we have shown that  $B \subseteq \neg A$  holds.

## Using the Contrapositive for a proof

Let's consider proving the same statement...

If  $A \cap B = \emptyset$ , then  $B \subseteq \neg A$ .

The contrapositive of this statement is the following:

if  $\neg(B \subseteq \neg A)$  then  $\neg(A \cap B = \emptyset)$ .

If we prove this statement, we have also proven the original, since a statement and its contrapositive are equivalent.

So, we will assume that  $\neg(B \subseteq \neg A)$ . Under this assumption we must show that  $A \cap B \neq \emptyset$ .

If we have that  $\neg(B \subseteq \neg A)$ , then there exists an element  $x$  such that  $x$  is IN  $B$  but NOT IN the complement of  $A$ . Symbolically we have:

$$\exists x \mid x \in B \wedge x \notin \neg A$$

If we have an element  $x \notin \neg A$ , by definition of complement, we must have that  $x \in A$ . So we have

$$\exists x \mid x \in B \wedge x \in A.$$

But, this means that  $x \in A \cap B!!!$  From that we can conclude that  $A \cap B \neq \emptyset$ , as desired.

## Proof by Contradiction

Most of this will look quite similar to the proof before. In essence there is little difference between a proof by contradiction and a proof of the contrapositive. Here's the basic idea:

Start out as if you were directly proving the statement:

- 1) Assume the "if" part of the statement.
- 2) Next, in contradiction to what you are trying to prove, assume the OPPOSITE of the "then" clause.
- 3) Start with the assumption in step 2 and make logical deductions
- 4) Eventually, you should reach a conclusion that is absurd, or directly contradicts the first assumption. Either way, this

**shows that the first two assumptions can NOT be true together, which, in essence, proves the statement.**

**Let's go back to our favorite example :)**

**If  $A \cap B = \emptyset$ , then  $B \subset \neg A$ .**

**So we first assume that  $A \cap B = \emptyset$ .**

**Next we will also assume that  $B \subseteq \neg A$  is false.**

**If we have  $\neg(B \subseteq \neg A)$ , then we must have an element  $x$  that is in  $B$  that is NOT in the complement of  $A$ . Symbolically we have:**

$$\exists x \mid x \in B \wedge x \notin \neg A$$

**As before, this is equivalent to...**

$$\exists x \mid x \in B \wedge x \in A.$$

**But this is impossible – it contradicts our assumption  $A \cap B = \emptyset$ . Thus, we must have made a mistake in our proof. Every step we took was logical EXCEPT for our assumption. Thus, this is what must be wrong...we can conclude that the opposite of it, is indeed true. This proves that**

**If  $A \cap B = \emptyset$ , then  $B \subset \neg A$ .**

## Couple More Examples

Prove that  $(B \subseteq C) \Rightarrow (B - A) \subseteq (C - A)$ .

To prove an if-then statement, we assume the premise is true. With that given information, we must show that the conclusion always follows.

Since  $B \subseteq C$ , we know for an arbitrary  $x \in B$ , that  $x \in C$ .

Now, we must prove  $(B - A) \subseteq (C - A)$  holds under this assumption.

To prove  $(B - A) \subseteq (C - A)$ , we must prove for an arbitrary  $x$ , that if  $x \in B - A$ , then  $x \in C - A$ .

To prove this statement, we assume that for an arbitrary  $x$ ,  $x \in B - A$ . By definition, this means that  $x \in B \wedge x \in \neg A$ . Since we know that  $B \subseteq C$ , we can deduce that  $x \in C$ . So now we know that  $x \in C \wedge x \in \neg A$ , thus  $x \in C - A$ , proving the assertion.

-----  
Prove  $A \neq B \Rightarrow (A \cup B) \neq (A \cap B)$

if  $A \neq B$  then we know

$\exists x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$ , since the definition of equality of sets is

$\forall x [(x \in A \wedge x \in B) \vee (x \notin B \wedge x \notin A)]$ .

**Without loss of generality, let  $x$  be an arbitrary element such that  $x \in A \wedge x \notin B$ .**

**If this is the case, then we know that  $x \in (A \cup B)$ , since  $x \in A$ .**

**However, we also know that  $x \notin (A \cap B)$ , since  $x \notin B$ .**

**Since there exists an element  $x$  contained in the set  $(A \cup B)$ , but that is NOT contained in the set  $(A \cap B)$ , we can conclude that the two sets can not be equal, since the definition for equality says for each element contained in one set must be contained in the other.**