

Set Theory

A set is a collection of objects, or *elements*. Typically, the type of all the elements in a set is the same. (For example, all the elements in a set could be integers.) However, it is possible to have different types of elements in a set. (An analogy for this is that usually a bookbag contains just books. But sometimes it may contain other elements such as pencils and folders as well.)

We have two usual methods of denoting the elements in a set:

- 1) Explicitly list the elements inside of a set of curly braces($\{\}$), as follows: $\{1, 2, 3, 4\}$
- 2) Give a description of the elements in a set inside of a set of curly braces as follows: $\{2x \mid x \in \mathbb{N}\}$.

In order to understand the second method, we must define the various symbols that are used in this notation. Here is a list of the symbols we will be using:

- $|$ - translates to “such that”
- \in - “is an element of”
- \subset - “is a proper subset of”
- \subseteq - “is a subset of”

Now we have to define what a subset is. A subset is also a set. So, if we have sets A and B, $A \subseteq B$ if for all $x \in A$, $x \in B$.

In layman’s terms, a set A is a subset of a set B, if all the elements in the set A also lie in the set B.

Note: $A \subset B$ iff $A \subseteq B \wedge A \neq B$.

We still have to define what $\{ 2x \mid x \in \mathbb{N} \}$ really means. Here it is in English:

“The set of all numbers of the form $2x$ such that x is an element of the natural numbers.” (Note: The set \mathbb{N} denotes the natural numbers, or the non-negative integers, according to the book.)

So, the set above could also be listed as $\{0, 2, 4, 6, \dots\}$

Now that we have gotten that out of the way, let’s talk about the empty set (\emptyset). The empty set is a set with no elements in it. In our standard notation, we could denote it as $\{\}$. It is also very common to use \emptyset , to denote the empty set.

It’s important to denote that the following are not equal:

\emptyset , $\{0\}$, and 0 .

The first two are sets, while the third is an element. However, the empty set has no elements while $\{0\}$ contains one element, zero.

Typically, sets will be denoted by uppercase letters. There are some other sets we should be familiar with since they come up so often. Here they are:

$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ (the set of integers)

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ (the set of non-negative integers)

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ (the set of positive integers)

$\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z} \wedge b \neq 0\}$

$\mathbb{R} =$ the set of real numbers...

Also, one last definition... $|A|$ for a set A is known as the “cardinality” of A , which equals the number of elements in A .

Set Operators

Now we are ready to discuss set operators. We can use several operators on existing sets to define new ones. The first two operators are binary operators, union and intersection. In each of these examples, let A and B be sets.

$$\text{union}(\cup) : A \cup B = \{ x \mid x \in A \vee x \in B \}$$

$$\text{intersection}(\cap) : A \cap B = \{ x \mid x \in A \wedge x \in B \}$$

$$\text{complement}(\neg) : \neg A = \{ x \mid x \notin A \}$$

$$\text{relative complement}(-) : B - A = \{ x \mid x \in B \wedge x \notin A \}$$

In English, the union of two sets contains all elements in either set and the intersection of two sets contains all elements in BOTH sets.

To define the complement, we must define what an universe is. For each set, there is a possible set of elements. This possible set of elements is known as the universe. Typically, you will be told what the universe is for each problem. (Most of the time it is the set of integers, or real numbers...) The complement of a set contains all the elements in the universe that are NOT in the set itself.

You can think of relative complement as the subtraction between two sets. $B - A$ refers to a set that subtracts out all the elements from A out of B. Now if a particular element of A wasn't in B to begin with, there's no need to take it out of B at all... Also, an identity that we can use is that $B - A = B \cap \neg A$.

Equality of Sets

There are three essentially different ways that we can show two sets to be equal. The first two are going to be analogous to methods used in logic.

1) Use the laws of set theory.

2) Use the table method.

I will go through the laws of set theory, showing you why they work through something called a Venn Diagram.

Then we will consider working through showing $A = (A - B) \cup (A \cap B)$.

Set Laws

1. $\neg\neg A = A$

Law of Double
Complement

2. $\neg(A \cup B) = \neg A \cap \neg B$
 $\neg(A \cap B) = \neg A \cup \neg B$

De Morgan's Laws

3. $A \cup B = B \cup A$
 $A \cap B = B \cap A$

Commutative Laws

4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Associative Laws

5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributive Laws
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. $A \cup A = A, A \cap A = A$ **Idempotent Laws**

7. $A \cup \emptyset = A, A \cap U = A$ **Identity Laws**

8. $A \cup \neg A = U, A \cap \neg A = \emptyset$ **Inverse Laws**

9. $A \cup U = U, A \cap \emptyset = \emptyset$ **Domination Laws**

10. $A \cup (A \cap B) = A$ **Absorption Laws
$A \cap (A \cup B) = A$**

Now, using these laws we can prove that $A = (A - B) \cup (A \cap B)$, for all sets A and B.

$$\begin{aligned} (A - B) \cup (A \cap B) &= (A \cap \neg B) \cup (A \cap B), \text{ defn of } - \\ &= A \cap (\neg B \cup B), \text{ distributive law} \\ &= A \cap U, \text{ Inverse law} \\ &= A, \text{ Identity law} \end{aligned}$$

Table Method

Another way to think of showing set equality is to look at a Venn Diagram and consider each possible position an arbitrary element can be located.

In essence, we know that two sets A and B are equal if $x \in A \Leftrightarrow x \in B$, for an arbitrary element x. What we can do is an exhaustive search of all the “places” an element x could like in a Venn diagram. If, in each situation, the above bidirectional implication holds, then we have proven the statement for an arbitrary x, proving the two sets to be equal.

Let's look at an example showing that

$$A = (A - B) \cup (A \cap B)$$

A	B	A - B	A \cap B	(A - B) \cup (A \cap B)
0	0	0	0	0
0	1	0	0	0
1	0	1	0	1
1	1	0	1	1

Finally, a third way to show that two sets A and B are equal is to prove that $A \subseteq B$ AND $B \subseteq A$.

Let's use this idea to show once again that

$$A = (A - B) \cup (A \cap B).$$

Part 1: Show that $A \subseteq (A - B) \cup (A \cap B)$.

Consider an arbitrary element $x \in A$. We must show that $x \in (A - B) \cup (A \cap B)$.

Now, consider the situation that $x \notin B$. In this particular situation, by definition, we know that $x \in (A - B)$, and thus **MUST BE** an element of $(A - B) \cup (A \cap B)$, since $(A - B)$ is a subset of $(A - B) \cup (A \cap B)$.

So, the only situation we have not proven the statement to be true for is if $x \in B$. But, if this is the case, then by definition, we know that $x \in A \cap B$, which implies that $x \in (A - B) \cup (A \cap B)$, as desired, proving part 1.

Part 2: Show that $(A - B) \cup (A \cap B) \subset A$.

It suffices to show $((A - B) \subset A) \wedge ((A \cap B) \subset A)$.

One requirement for $x \in A - B$ is that $x \in A$, proving the first part.

Furthermore, one requirement for $x \in A \cap B$ is that $x \in A$, proving the second part of the and, thus we have shown that $(A - B) \cup (A \cap B) \subset A$.

Here is a more difficult example. See if you can show that these two sets are equal using either of the methods I have presented thus far.

$$\neg(\neg((A \cup B) \cap C) \cup \neg B) = (B \cap C)$$