Composition of Relations

In math class, given two functions f(x) and g(x), you probably had to figure out the composition of the functions, which is denoted either by f(g(x)) OR f°g(x).

Basically, the way this worked is that you "plugged in" your original x into one function, THEN you used the "answer" that you got from that function to "plug in" to the second function. And the order in which you did it mattered.

The same will be true of the composition of two relations. Here is the formal definition of the composition of two relations R and S, where $R \subseteq A \times B$, $S \subseteq B \times C$:

$$S \circ R = \{ (a,c) \mid a \in A \wedge c \in C \wedge (\exists b \mid (a,b) \in R \wedge (b,c) \in S) \}$$

Notice that this is extremely similar to the definition of function composition you learned in high school. Basically, when you compose the relations R and S, you get a third relation which relates elements from the set A to the set C, as long as the "answer" from relation R can be the input for relation S.

We can use a directed graph again. Consider this example:
A = { ABC, NBC, CBS, FOX, HBO}
B = { NYPD Blue, Simpsons, Letterman, ER, X-Files,
Dennis Miller Show, Monday Night Football}
C = { Dennis Miller, Marge, Rick Schroeder, Gillian Anderson,
Noah Wyle}
R = {(ABC, NYPD Blue), (NBC, ER), (CBS,Letterman),
(HBO, Dennis Miller Show), (FOX, X-Files) }
S = { (MNF, Dennis Miller), (Simpsons, Marge),
(ER, Noah Wyle), (Party of 5, Neve Campbell)
(D. Miller Show,Dennis Miller),(NYPD, Rick Schroeder) }

Theorems about Relation Composition

If $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq C \times D$, then we have the following:

$(T \circ S) \circ R = T \circ (S \circ R)$

Essentially, when doing multiple relation composition, associativity is preserved.

First of all, we see that both sides define a relation over the set A x D. Next, we have to prove that both define the same relation over that set.

Formally, if we break down the definition, we have:

$(T \circ S) \circ R = \{(a, d) | a \in A$ and $d \in D$, and for some $b \in B$, $(b, d) \in T \circ S$ and $(a, b) \in R\}$,

Since $(b, d) \in T \circ S$ means $(b, c) \in S$ and $(c, d) \in T$ for some $c \in C$, by definition of $T \circ S$, the relation $(T \circ S) \circ R$ consists of ordered pairs $(a, d) \in A \times D$ such that for some $b \in B$ and some $c \in C$, $(a, b) \in R$, $(b, c) \in S$ and $(c, d) \in T$.

If we break down the definition of $T \circ (S \circ R)$ in a similar manner, we will get the exact same thing. Similarly, using the directed graph of the situation will lead to the same conclusion.

Note: that if a and b are elements and R is a relation, the statement $(a, b) \in R$ may be written as aRb.

Let $R \subseteq A \times B$, $S \subseteq B \times C$, and $T \subseteq B \times C$ denote 3 binary relations.

Then we have the following:

(1) $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$
(2) $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$. (Usually, this is a proper subset.)

Here is why the first one holds:

First, plug into the definition of $(S \cup T) \circ R$:

$(S \cup T) \circ R$:

$= \{(a, c) | a \in A \wedge c \in C \wedge \exists b \in B | aRb \wedge (b, c) \in S \cup T\}$

$= \{(a, c) | a \in A \wedge c \in C \wedge \exists b \in B | aRb \wedge ((b, c) \in S \vee (b, c) \in T)\}$ (Definition of $\cup$)

$= \{(a, c) | a \in A \wedge c \in C, \wedge \exists b \in B \mid ((aRb \wedge bSc) \vee (aRb \wedge bTc))\}$ (Distributive property)

$= \{(a, c) | a \in A \wedge c \in C \wedge \exists b \in B \mid aRb \wedge bSc\} \cup$
$\{(a, c) | a \in A \wedge c \in C \wedge \exists b \in B \mid aRb \wedge bTc\}$

$= (S \circ R) \cup (T \circ R)$, definition of $(S \circ R)$ and $(T \circ R)$.

Every single one of these steps were equality (bidirectional) steps, thus the equality between the sets holds. Each step would be valid working "backwards" to show that the RHS is a subset of the LHS.

$(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$

To prove this statement, we must show that an arbitrarily chosen element of the LHS is also an element of the RHS. Let (a, c) be an arbitrarily chosen element of $(S \cap T) \circ R$.

Since $(a, c) \in (S \cap T) \circ R$ there exists $b \in B$, such that aRb and $(b, c) \in (S \cap T)$, by the definition of $\circ$. Thus, $(b, c) \in S$ and $(b, c) \in T$, by the definition of $S \cap T$. Therefore, since aRb, so $(a, c) \in (S \circ R)$ and $(a, c) \in (T \circ R)$, by the definition of $\circ$. Thus, $(a, c) \in (S \circ R) \cap (T \circ R)$, and (2) is proved.

As an exercise, I want you to find a counterexample to the claim that $(S \circ R) \cap (T \circ R) \subseteq (S \cap T) \circ R$.

Here is a counter example:
$A = \{1,2\}$ $B = \{a,b\}$ $C = \{x,y\}$
$R = \{(1,a), (1,b)\}$
$S = \{(a,x)\}$
$T = \{(b,x)\}$

Here we have $S \cap T = \varnothing$, so $(S \cap T) \circ R = \varnothing$.
But, $S \circ R = \{(1,x)\}$ and $T \circ R = \{(1,x)\}$
so $(S \circ R) \cap (T \circ R) = \{(1,x)\}$, proving the statement false.

Intuitively, the idea here is that relations S and T provide different pathways for 1 to be related to x. From the graph point of view of relations, we can have different paths that reach from 1 to x (different set of roads).

Prove or disprove: If $S \circ R = T \circ R$, then $S = T$.

This statement is also false. Simply use the counter example given in the problem above to validate this claim.

Examples of Identifying Properties in Relations

Is the following relation reflexive, irreflexive, symmetric, antisymmetric, or transitive? $R = \{(a,b) \mid a,b \in Z^+ \wedge$ a, 2a, and b are side lengths of a triangle$\}$ Note: For all triangles, the sum of the lengths of any two sides must exceed the length of the third side.

Reflexive? No – because $(a,a) \notin R$, this is because a triangle can not have sid lengths, a, a and 2a.

Irreflexive? Yes – the previous argument holds for all positive integers a.

Symmetric? No – $(a, 2a) \in R$, since we can have a triangle with side lengths a, 2a and 2a. However, $(2a, a) \notin R$ because we can not have a triangle with side lengths 2a, 4a and a.

Antisymmetric? Yes – If we have $a \geq b$, then we have $(a,b) \notin R$. To prove this, consider a forming a triangle with side lengths a, 2a, and b. We know that we must have $a+b > 2a$ for a triangle to be formed. BUT, $a+b \leq a+a = 2a$, which means that a+b is NOT greater than 2a. Thus, in this situation, we have $(a,b) \notin R$. Thus, for any element $(a,b) \in R$, we must have $a < b$. For each of these elements, we can guarantee that $(b,a) \notin R$ since $b > a$.

Transitive? No – $(a, 2a) \in R$ as shown above, and we also know that $(2a, 4a) \in R$, by a similar analysis. But, we can show that $(a,4a) \notin R$ because a triangle can not have side lengths a, 2a and 4a.

Consider the following relation:

$R = \{ (a,b) \mid a \in Z^+ \wedge b \in Z^+ \wedge ab = c^2$ for some positive integer c$\}$

Prove that it is an equivalence relation.

Reflexive? Yes – $(a,a) \in R$ because $a^2 = c^2$, when c is equal to a, a positive integer.

Symmetric? Yes – if $(a,b) \in R$, we must show that $(b,a) \in R$. If $(a,b) \in R$, we know that $ab= c^2$, for a positive integer c. But, we know that multiplication is commutative, so $ba = c^2$. Thus, $(b,a) \in R$.

Transitive? Yes - if $(a,b) \in R$ and $(b,c) \in R$, we must show that $(a,c) \in R$. We know that if $(a,b) \in R$, then $ab = d^2$, for some positive integer d. Furthermore, if $(b,c) \in R$, then $bc = e^2$, for some positive integer e.

$ab = d^2$
$bc = e^2$
Multiplying these equations, we find

$ab^2c = (de)^2$
$ac = (de/b)^2$
Now, if we can show that de/b is an integer, we will have shown that $(a,c) \in R$.

Technically speaking, this is difficult to show. But, we know that if a number is not a perfect square, its square root is irrational. But, from the above we have the square root of ac is de/b, a rational quantity. Thus, we must have that ac is a perfect square, which means de/b is an integer.

Closures for Binary Relations

Let $R \subseteq A \times A$ denote a binary relation. The following relations defined over A are called *closures*:

The *reflexive closure* of R is $r(R) = R \cup \{(a, a) \mid a \in A\}$.

The *symmetric closure* of R is $s(R) = R \cup R^{-1}$.

The *transitive closure* of R is $t(R) = R \cup R^2 \cup R^3 \cup \ldots$, where $R^2 = R \circ R$, $R^3 = R^2 \circ R$, etc., where $\circ$ denotes relation composition. Thus, $(a, b) \in t(R) \Leftrightarrow (a, b) \in R^n$, for some $n \geq 1 \Leftrightarrow$ there exist $a_1, a_2, \ldots, a_n \in A$, $a_n = b$, for some $n \geq 1$, such that $(a, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n) \in R$, i.e., there exists a direct path of n edges connecting a to b in the digraph for the relation R.

It can easily be seen that the names of these closures are justified in that for any binary relation R, $r(R)$ is reflexive, $s(R)$ is symmetric, and $t(R)$ is transitive. Also, we can define the composition of these closures, e.g., $tr(R) = t(r(R))$, $rs(R) = r(s(R))$, etc.

Let's look at an example. Let A = {1, 2, 3, 4, 5} and let R = {(1,2), (2,3), (4,4), (4,5)} then we have the following:

r(R) = {(1,1), (2,2), (3,3), (5,5), (1,2), (2,3), (4,4), (4,5)}
s(R) = {(1,2), (2,1), (2,3), (3,2), (4,4), (4,5), (5,4)}
t(R) = {(1,2), (2,3), (4,4), (4,5), (1,3)}

Let's look at some relation problems dealing with closures:

1) If R is transitive, then *r(R)* is transitive.

Let R be a transitive relation. Now, consider r(*R)*. The definition for transitivities says that A relation R is transitive if aRb and bRc implies aRc. Let's consider this statement for arbitrary values a,b and c where the three are distinct, and when they are not.

If the three are distinct, we know that only if (a,b)$\in$R and (b,c)$\in$R would (a,b)$\in$*r(R)* and (a,b)$\in$*r(R)*. But in this situation, we have (a,c)$\in$R, which means that (a,c) $\in$*r(R)*.

Now, we must consider the cases where a, b and c are not necessarily distinct. There are three of these cases :

1) a=b, but b$\neq$c
2) a$\neq$b but b=c
3) a=b=c

In case 1, if we have that (a,a)$\in$r(R) and (a,c)$\in$r(R), we know that transitivity holds since transitivity implies that (a,c)$\in$r(R), but this was part of the premise. Cases 2 and 3 also boil down the same way. Essentially, if either a=b or b=c, if the premise of transitivity is true, then the conclusion MUST BE.

2) If *R* is symmetric and $R \subseteq T$, then *T* is symmetric.

This is false. Consider the following counter example:

R = {(a,b),(b,a)} and T = {(a,b),(b,a),(b,c)}

3) If $R \neq T$, then $s(R) \neq s(T)$

This is false as well. Consider $R = \{(1,2), (2,1)\}$ and $T = \{(1,2)\}$.

4) $(R \cup T)^{-1} = R^{-1} \cup T^{-1}$

This is true. To show this, we must show that the LHS is a subset of the RHS and vice vera.

Consider an arbitrary element (a,b) of $(R \cup T)^{-1}$. This means that (b,a) MUST BE an element of $R \cup T$. This leaves us with two possibilities. Either we have $(b,a) \in R$ or $(b,a) \in T$. Consider the first case. In this case we must have that $(a,b) \in R^{-1}$, which certainly means that $(a,b) \in R^{-1} \cup T^{-1}$. Now we must consider the other case - $(b,a) \in T$. This means that $(a,b) \in T^{-1}$, which also shows that $(a,b) \in R^{-1} \cup T^{-1}$. From this we can conclude that $(R \cup T)^{-1} \subset R^{-1} \cup T^{-1}$

Now, we must show the other side that $R^{-1} \cup T^{-1} \subset (R \cup T)^{-1}$. Consider an arbitrary element (a,b) of $R^{-1} \cup T^{-1}$. There are two possibilities here. Either we have $(a,b) \in R^{-1}$ or $(a,b) \in T^{-1}$. Consider the first. In this case, $(b,a) \in R$. If this is the case then certain $(b,a) \in R \cup T$. But if this is the case, then we must have $(a,b) \in (R \cup T)^{-1}$. Our other case to consider is $(b,a) \in T$. Similarly, we know that $(b,a) \in R \cup T$ and $(a,b) \in (R \cup T)^{-1}$. Thus, either way, we have shown that any arbitrary element of $R^{-1} \cup T^{-1}$ must also be in the set $(R \cup T)^{-1}$. This proves that $R^{-1} \cup T^{-1} \subseteq (R \cup T)^{-1}$. Combining this with the first half of the proof, we find that $(R \cup T)^{-1} = R^{-1} \cup T^{-1}$.

5) Prove that $t(R) \subseteq R \cup (R \circ t(R))$.

Consider an arbitrary element $(a,b) \in t(R)$. We must show that $(a,b) \in R \cup (R \circ t(R))$.

If we have that $(a,b) \in R$, we are done.

Otherwise, we know that $(a,b) \notin R$. This must mean that $(a,b) \in R^n$, where n is an integer greater than 1.

But, consider the set $R \circ t(R)$. This is equal to $R \circ (R \cup R^2 \cup R^3 \cup \ldots)$ Applying each function composition, we find that this is

$(R^2 \cup R^3 \cup R^4 \cup \ldots)$

Clearly, R^n, where n is an integer greater than 1 is a subset of the set $(R \circ t(R))$ above. Thus, since $(a,b) \in R^n$, we must have that $(a,b) \in (R \circ t(R))$, proving the original assertion.