

## Mathematical Functions

In mathematics, a function is an equation where you “plug in” a value, and get an “answer” so to speak. In particular, whenever you plug in a particular value, you must get a **SINGLE** answer. (You should also get the same answer always.) Functions graphed on the x-y plane have to pass the vertical line test.

Now, in discrete mathematics, we will be using functions a bit differently & we will also coin a new term “relation”. In particular, a function is a specific type of relation.

In standard high school mathematics, we typically deal with functions of one variable. We always graph a function of the form  $y=f(x)$ , where the left hand side is entirely dependent on  $x$ . Depending on what the function  $f(x)$  is, there is always a set of values that are **VALID** to “plug” in to the equation. This set is the domain. Similarly, the “answer” you get out of the function will always lie in a particular set. This set is the range.

The problem with using standard functions for discrete mathematics is that many are defined for all real numbers. Namely, it would be nice if we could list every value in the domain of some function. But, we **CAN NOT** list out each real number. (We can list out each integer however...)

The basis of functions and relations in discrete mathematics is the idea that values of a domain and range should be subsets of a set that can be listed, such as the integers, color, etc.

As we go through different things, I will make analogies to mathematical functions, so you can see the similarities between these and the functions and relations for discrete mathematics.

## Relations

**A relation is something that relates one set of values to another set of values. Sometimes the relationship that is specified between sets is meaningful, other times it is not.**

**In general, a relations are defined in the following manner:**

**A relation  $R$  defined over sets  $A$  and  $B$  is a subset of  $A \times B$ . Thus, we have  $R \subseteq A \times B$ . This is known as a binary relation, because it relates elements between two sets.**

**Consider this example:**

**Let  $A = \{\text{Orange Juice, Cranberry Juice, Coke}\}$  and  
 $B = \{\text{Rum, Vodka, Peach Schnapps}\}$**

**If you had some modicum of taste, we could define a relation Cocktails as follows:**

**Cocktails =  $\{ (\text{Orange Juice, Vodka}), (\text{Cranberry Juice, Vodka}), (\text{Coke, Rum}), (\text{Orange Juice, Peach Schnapps}) \}$**

**Of course, if you do not have any standards, we could have up to 9 pairs listed in our relation for Cocktails.**

**Graphically, we could use a directed graph to represent this information as follows:**

Of course, you can see there are some restrictions with only being able to define binary relations. For example, even if we extended our sets  $A$  and  $B$  from the previous example to provide for a fully stocked bar, we **STILL** could not define a relation that would include a Long Island Ice Tea. (For any one not familiar with this drink, it contains 4 or 5 elements from an extended version of set  $B$ .)

Thus, we should define relations between more than two items. In general, we can define an  $n$ -ary relation as follows:

An  $n$ -ary relation  $R$  over sets  $A_1, A_2, A_3, \dots, A_n$  is a subset of the cartesian product  $A_1 \times A_2 \times A_3 \dots \times A_n$ . The degree of this relation  $R$  is  $n$ .

Now, we could define a relation on  $A \times A \times B \times B \times B \times B$  that would include a Long Island Ice tea as an element of it.

Of course, it is probably more typical that an  $n$ -ary relation be comprised of several different sets, but there is no rule against defining a relation using the same set repeatedly, as we have done above.

Also, we can denote an  $n$ -ary relation using a table as follows:

Mixer 1	Mixer 2	Liquor 1	Liquor 2	Liquor 3	Liquor 4
Coke	Sour Mix	Vodka	Tequila	Rum	Gin
...	...	...	...	...	...

## Definitions for Binary Relations over $A \times A$

A majority of the binary relations we will be dealing with are a subset of the Cartesian product of a particular set with itself.

If we have  $R \subseteq A \times A$ , then we have the following definitions:

- 1)  $R$  is reflexive if  $\forall a \in A, (a,a) \in R$ .
- 2)  $R$  is irreflexive if  $\forall a \in A, (a,a) \notin R$ .
- 3)  $R$  is symmetric if  $\forall a \in A, aRb \Rightarrow bRa$
- 4)  $R$  is anti-symmetric if  $aRb \wedge bRa \Rightarrow a=b$ .
- 5)  $R$  is transitive if  $aRb \wedge bRc \Rightarrow aRc$ .

Consider the following relation  $R$  defined over  $\{a, b, c\}$ :

$$R = \{ (a,b), (a,c), (b,a), (b,c), (c,c) \}$$

$R$  is not reflexive since  $(b,b) \notin R$

$R$  is not irreflexive since  $(c,c) \in R$

$R$  is not symmetric since we have  $(a,c) \in R$ , but  $(c,a) \notin R$ .

$R$  is not anti-symmetric since  $(a,b) \in R$  and  $(b,a) \in R$ .

$R$  is not transitive since  $(b,a) \in R, (a,b) \in R$ , but  $(b,b) \notin R$ .

Now, I will show you some examples of more meaningful relations that actual have some of these properties.

Consider a relation  $R$  over the set {jelly, bread, ham} that is defined as foods that go well together. The relation could be:

$R = \{ (jelly, jelly), (bread, bread), (ham, ham), (jelly, bread), (bread, jelly), (ham, bread), (bread, ham) \}$

This relation is reflexive since for each element  $a$ ,  $(a,a) \in R$ . Essentially, we can mix anything with itself and it'll still be edible.

This relation is also symmetric. The reason for this is that if we can mix one food first with a second food, then we can ALSO mix the second food with the first. Symbolically, for each pair for related items  $a,b \in A$  such that  $aRb$ , we also have  $bRa$ .

This relation is NOT transitive. Essentially, the way we have it defined, we can have jelly with bread, and bread with ham, but we CAN NOT eat jelly with ham...and that violates transitivity.

Now consider this example that is more mathematical.  $R \subseteq Z^+ \times Z^+$  as follows:

$$R = \{(a,b) \mid a \in Z^+ \wedge b \in Z^+ \wedge (\exists n \mid n \in Z^+ \wedge a = bn) \}$$

In English, this reads, the relation defined over all positive integers such that the first integer is a multiple of the second integer.

This is reflexive because for all positive integers  $a$ , we have  $aRa$ , since  $a=1*a$ .

This is anti-symmetric. If we have  $aRb$  and  $bRa$ , that means that  $a = n_1*b$  for some integer  $n_1$ , and that  $b = n_2*a$  for some integer  $n_2$ . Substitute in for  $b$  in the first equation to get

$$a = n_1*n_2*a$$

Divide both sides by  $a$  to get  $1 = n_1*n_2$ . We know if  $n_1, n_2 \in Z^+$ , then both of them MUST BE 1, hence  $a=b$ .

This relation is transitive. Consider the situation where  $aRb$  and  $bRc$ . Then we have the following:

$$a = nb, \text{ for some positive integer } n$$

$$b = mc, \text{ for some positive integer } m$$

Thus,  $a = n(mc) = (nm)*c$  which implies that  $aRc$ .

Any relation that is reflexive, anti-symmetric AND transitive is known as a partial ordering relation. A partial ordering relation is one that can “compare” elements together in a hierarchical way. (One element is at least as high as another one, on the hierarchy.)

## Equivalence Relations

An equivalence relation is one that is similar to a partial ordering relation defined above, but different in one critical aspect: it must be symmetric instead of anti-symmetric. In particular, an equivalence relation gives a way to group like elements in a set together.

Consider this standard example used in CS classes when introducing an if statement:

Define a relation over the set  $A = \{0,1,\dots,100\}$  as follows:

$$R = \{(a,b) \mid a \in A \wedge b \in A \wedge (\lfloor a/10 \rfloor = \lfloor b/10 \rfloor)\}$$

In essence, this relates elements together if they lie within the same grade range(0-9, 10-19, ..., 90-99, 100).

We can see how this is reflexive – for all elements  $a$  in  $A$  we have  $\lfloor a/10 \rfloor = \lfloor a/10 \rfloor$ .

Clearly this is symmetric too. If we have  $(\lfloor a/10 \rfloor = \lfloor b/10 \rfloor)$  then we have  $(\lfloor b/10 \rfloor = \lfloor a/10 \rfloor)$ . Hence if  $aRb$ , then  $bRa$ .

Finally, this is also transitive. If  $(\lfloor a/10 \rfloor = \lfloor b/10 \rfloor)$  and  $(\lfloor b/10 \rfloor = \lfloor c/10 \rfloor)$ , we always have that  $(\lfloor a/10 \rfloor = \lfloor c/10 \rfloor)$ .

**Try to show that modular equivalence is a equivalence relation over the positive integers. Remember that a modular equivalence is the following:**

$$a \equiv b \pmod{m} \text{ iff } m \mid (a-b).$$

**Thus the relation defined is the following:**

$$R = \{ (a,b) \mid a \equiv b \pmod{m}, \text{ where } m \text{ is an arbitrary pos. int.} \}$$

**The relation is clearly reflexive since for all positive integers  $a$ , we have  $m \mid a - a$ .**

**The relation is also symmetric since if we have  $aRb$ , that means that we have  $m \mid a - b$ . But, if  $m \mid a - b$ , we have that  $m \mid b - a$ . (Formally, you'd say that  $mc = a - b$ , for some integer  $c$ , then we have  $m(-c) = b - a$ , hence  $m \mid (b - a)$ .)**

**Finally, if we have  $aRb$  and  $bRc$ , then we know the following:**

$$m \mid a - b$$

$$m \mid b - c$$

**$m$  must divide the sum of these, hence we have**

$$m \mid (a - b) + (b - c), \text{ so } m \mid a - c, \text{ but this implies that}$$

$$a \equiv c \pmod{m}, \text{ thus we have shown that } aRc, \text{ as desired.}$$



## Equivalence Classes

Now, each element that is related to each other in an equivalence relation is said to be in the same equivalence class. More formally, we have the following:

For an equivalence relation  $R$  over a set  $A$ , an equivalence class for an element  $x$  is defined as follows:

$$[x] = \{ a \mid a \in A, \text{ and } aRx \}$$

Next, we can define a partition of the set  $A$  based of the relation  $R$  as follows:

- 1) if  $x \in A \wedge y \in A$ , either  $[x] = [y]$  OR  $[x] \cap [y] = \emptyset$ .
- 2) if  $x \in A$ ,  $x \in [x]$

We can show that all equivalence relations  $R$  over a set  $A$  adhere to the restrictions above.

We can prove 1, as follows:

If  $[x] \cap [y] = \emptyset$ , we are done. Now consider the other case that  $[x] \cap [y] \neq \emptyset$ . That means that we have an element  $z$  of  $A$  such that  $z \in [x]$  AND  $z \in [y]$ . This means that  $zRx$  and  $zRy$ . But, by the symmetry of  $R$  we have  $xRz$ . Then, due to transitivity, we have  $xRy$ . Now, we can prove that  $[x] \subseteq [y]$ . Consider an arbitrary  $a \in [x]$ . This means we have  $aRx$ , but we also have  $xRy$ . Hence, by transitivity, we have  $aRy$ , so  $a \in [y]$ , proving that  $[x] \subseteq [y]$ . Similarly, we can prove  $[y] \subseteq [x]$ . Thus, we have  $[x] = [y]$ .

We always have  $x \in [x]$ , since  $R$  is reflexive.

**If  $R$  is an equivalence relation defined over a set  $A$ , we will let  $A / R$  denote the set of these equivalence classes. Formally,**

$$A / R = \{[x] \mid x \in A\}.$$

**Finally, we shall prove that any partition of a set such that each element is related to each other element in a partition is an equivalence relation.**

**Let  $R$  be the relation for a particular partition.**

**Clearly,  $R$  is reflexive, since each element in a single partition is related to itself, and each element belongs in a partition.**

**$R$  is also symmetric since if  $xRy$ , both  $x$  and  $y$  HAVE to be in the same partition, hence we must also have  $yRx$ .**

**Finally, if we have  $xRy$  and  $yRz$ , that must mean that  $x$ ,  $y$ , and  $z$  HAVE to be in the same partition. In particular,  $x$  and  $z$  are in the same partition, thus we also have  $xRz$ , proving that  $R$  is transitive as well.**

**Hence,  $R$  is an equivalence relation.**