Probability Questions (Taken from past IB HL Exams)

1) ('05 IB HL P1 #7) In an examination of 20 multiple-choice questions each question has four possible answers, only one of which is correct. Robert randomly guesses the answer to each question.

(a) Find his expected number of correct answers.
(b) Find the probability that Robert obtains this expected number of correct answers

2) ('05 IB HL P1 #10) In a survey of 50 people it is found that 40 own a television and 21 own a computer. Four do not own either a computer or television. A person is chosen at random from this group.

(a) Find the probability that this person owns both a television and computer.
(b) Given that this person owns a computer, find the probability that he also owns a television.

3) ('04 IB HL P1 #13) A desk has three drawers. Drawer 1 contains three gold coins, Drawer 2 contains two gold coins and one silver coing and Drawer 3 contains one gold coin and two silver coins. A drawer is chosen at random and from it a coin is chosen at random.

(a) Find the probability that the chosen coin is gold.
(b) Given that the chosen coin is gold, find the probability that Drawer 3 was chosen.

4) ('03 IB HL P1 #6) When a boy plays a game at a fair, the probability that he wins a prize is 0.25. He plays the game 10 times. Let X denote the total number of prizes he wins. Assuming that the games are independent, find (a) E(X) and (b) $P(X \leq 2)$.

5) ('02 IB HL P1 #7) The probability that it rains during a summer's day in a certain town is 0.2. In this town, the probability that the daily maximum temperature exceeds 25 degrees Celsius is 0.3 when it rains and 0.6 when it does not rain. Given that the maximum daily temperature exceeded 25 degrees Celsius on a particular summer's day, find the probability that it rained on that day.

6) ('02 IB HL P1 #9) When John throws a stone at a target, the probability that he hits the target is 0.4 He throws a stone 6 times. (a) Find the probability that he hits the target **exactly** 4 times. (b) Find the probability that he hits the target for the first time on his third throw.

7) ('01 IB HL P1 #11) Given that $P(X) = 2/3$, $P(Y \mid X) = 2/5$ and $P(Y \mid X') = 1/4$, find

(a) $P(Y')$ (b) $P(X' \cup Y')$.

8) ('00 IB HL P1 #7) In a game a player rolls a biased tetrahedral (four-sided) die. The probability of each possible score is shown below:

Score	1	2	3	4
Probability	1/5	2/5	1/10	x

Find the probability of a total score of six after two rolls.

9) ('00 IB HL P1 #12) The probability distribution of a discrete random variable X is given by

$$P(X = x) = k\left(\frac{2}{3}\right)^x, \text{ for } x = 0, 1, 2, \ldots$$

Find the value of k.

10) ('04 IB HL P2 #2(i)) Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.

(a) Write down the probability that Jack wins on his first throw.
(b) Calculate the probability that Jill wins on her first throw.
(c) Calculate the probability that Jack wins the game.

11) ('03 IB HL P2 #4) A business man spends X hours on the telephone during the day. The probability density function of X is given by

$$f(x) = \frac{1}{12}(8x - x^3), \text{ for } 0 \le x \le 2$$
$$= 0, \text{ otherwise}$$

(a) (i) Write down the integral whose value is E(X).
(ii) Hence evaluate E(X).

(b) (i) Show that the median, m, of X satisfies the equation $m^4 - 16m^2 + 24 = 0$.
(ii) Hence evaluate m.

(c) Evaluate the mode of X.

12) ('02 IB HL P2 #4) Two children, Alan and Belle, each throw two fair cubical dice simultaneously. The score for each child is the sum of the two numbers shown on their respective dice.

(a) (i) Calculate the probability that Alan obtains a score of 9.
(ii) Calculate the probability that Alan and Belle both obtain a score of 9.

(b) (i) Calculate the probability that Alan and Belle obtain the same score.
(ii) Deduce the probability that Alan's score exceeds Belle's score.

(c) Let X denote the largest number shown on the four dice.

(i) Show that $P(X \leq x) = \left(\frac{x}{6}\right)^4$, for x = 1,2,3,4,5,6

(ii) Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
P(X = x)	1/1296	15/1296				671/1296

(iii) Calculate E(X).

Solutions to Questions Above

1) ('05 IB HL P1 #7) (a) E(X) = np = 20(.25) = **5**. (b) P(X=5) = $\binom{20}{5}(.25)^5(.75)^{15} = .202$

2) ('05 IB HL P1 #10) Let A be the set of TV owners and B be the set of computer owners. Since 4 own neither, we have that $|A \cup B| = 46$. Also, $|A \cup B| = |A| + |B| - |A \cap B|$. Substituting the known values into this equation we can solve for $|A \cap B|$:

$$46 = 40 + 21 - |A \cap B|$$
$$|A \cap B| = 15.$$

(a) P(person owns both) = $\frac{15}{50} = \frac{3}{10}$. (b) P(person owns comp | owns TV) = $\frac{15}{21} = \frac{5}{7}$.

In both of these, we divide by the sample space. In part (a) this is all 50 people, in part (b) this is the 21 people who own TVs. For both questions the numerator is represented by people who own both.

3) ('04 IB HL P1 #13)

(a) Let A, B and C be the events that Drawer 1, 2 and 3 are chosen, respectively. Let G be the event that a gold coin is chosen. Then we have:

P(G) = P(A)*P(G | A) + P(B)*P(G | B) + P(C)*P(G | C)

$$= \frac{1}{3} x 1 + \frac{1}{3} x \frac{2}{3} + \frac{1}{3} x \frac{1}{3} = \frac{2}{3}$$

(b) P(C | G) = $\frac{P(C \cap G)}{P(G)} = \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{6}$

4) ('03 IB HL P1 #6) When a boy plays a game at a fair, the probability that he wins a prize is 0.25. He plays the game 10 times. Let X denote the total number of prizes he wins. Assuming that the games are independent, find (a) E(X) and (b) P(X≤ 2).

This is a binomial distribution with p = 0.25 and n=10.

(a) E(X) = np = 10(.25) = 2.5

(b) $P(X \le 2) = \binom{10}{0}(.75)^{10} + \binom{10}{1}(.25)(.75)^9 + \binom{10}{2}(.25)^2(.75)^8 \approx 0.526$

5) ('02 IB HL P1 #7) The probability that it rains during a summer's day in a certain town is 0.2. In this town, the probability that the daily maximum temperature exceeds 25 degrees Celsius is 0.3 when it rains and 0.6 when it does not rain. Given that the maximum daily temperature exceeded 25 degrees Celsius on a particular summer's day, find the probability that it rained on that day.

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   Rains 0.2          Temp>25   0.3
*--------------------*--------------------* (0.06)
|                    |
| No rain 0.8        | Temp<=25 0.7
|                    |
|                    *(0.14)
|  Temp>25  0.6
*--------------------* (0.48)
|
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| Temp <= 25 0.4 P(Rained|Temp>25) = $\dfrac{P(Rained \cap Temp > 25)}{P(Temp > 25)} = \dfrac{(0.2)(0.3)}{0.54} = \dfrac{1}{9}$

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|
*(0.32)
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P(Temp>25) = (0.2)(0.3) + (0.8)(0.6) = 0.54

6) ('02 IB HL P1 #9) When John throws a stone at a target, the probability that he hits the target is 0.4 He throws a stone 6 times. (a) Find the probability that he hits the target **exactly** 4 times. (b) Find the probability that he hits the target for the first time on his third throw.

This is another binomial distribution with p=0.4 and n=6.

(a) $\binom{6}{4}(0.4)^4(0.6)^2 = .13824$

(b) (0.6)(0.6)(0.4) = .144, since each of the first three throws is forced to be a particular outcome.

7) ('01 IB HL P1 #11)

$P(Y \mid X) = \dfrac{P(Y \cap X)}{P(X)} = \dfrac{P(Y \cap X)}{2/3} = \dfrac{2}{5}$, thus $P(Y \cap X) = \dfrac{4}{15}$

$P(Y \mid X') = \dfrac{P(Y \cap X')}{P(X')} = \dfrac{P(Y \cap X')}{1/3} = \dfrac{1}{4}$, thus $P(Y \cap X') = \dfrac{1}{12}$

(a) Thus, we have $P(Y') = 1 - P(Y) = 1 - (P(Y \cap X) + P(Y \cap X')) = 1 - \dfrac{4}{15} - \dfrac{1}{12} = \dfrac{13}{20}$

(b) $P(X' \cup Y') = 1 - P(X \cap Y) = 1 - \dfrac{4}{15} = \dfrac{11}{15}$

8) ('00 IB HL P1 #7) Let P(n) denote the probability of obtaining a sum of n after two rolls, and P(a,b) denote the probability of rolling a followed by rolling b. Then we have:

P(6) = P(2,4) + P(3,3) + P(4,2)
= (2/5)(3/10) + (1/10)(1/10) + (3/10)(2/5)
=1/4

9) ('00 IB HL P1 #12) Since the sum of all probabilities must be one, we have $\sum_{x=0}^{\infty} k(\frac{2}{3})^x = 1$, so $k\sum_{x=0}^{\infty}(\frac{2}{3})^x = 1$, $k(\frac{1}{1-2/3}) = 1$, thus $k = \frac{1}{3}$

10) ('04 IB HL P2 #2(i))

(a) $\frac{2}{3}$, since 4 out of 6 opening rolls wins.

(b) P(Jill wins on the 1st throw) = P(Jack gets 5 or 6)*P(Jill gets 1,-4) = $\frac{1}{3} x \frac{2}{3} = \frac{2}{9}$

(c) Let X be the probability that the person who rolls first wins. Then 1-X is the probability that the person who rolls second wins. Notice that X is the sum of the probability that Jack wins on the first roll plus the probability that he wins on a later roll.

Since the first part of this was determined in (a), we must only determine the probability that Jack wins on a later roll. In order for this to occur, he must roll a 5 or 6, followed by Jill taking a turn. When Jill takes her turn, her probability of winning is now X (since in essence, she is the first player in the game), and her probability of losing is 1-X. Thus the probability of Jack winning is this product. This leads us to the equation:

$$X = \frac{2}{3} + \frac{1}{3}(1 - X)$$
$$X = 1 - \frac{X}{3}$$
$$X = \frac{3}{4}$$

A more straight-forward way of solving the problem is as follows:

P(Jack wins on roll k) =
P(Jack and Jill roll 5 or 6 for k-1 turns)*P(Jack rolls a 1,2,3 or 4)=
$(\frac{1}{3} x \frac{1}{3})^{k-1} x \frac{2}{3} = \frac{1}{9}^{k-1} x \frac{2}{3}$

P(Jack wins) = $\sum_{k=1}^{\infty}(\frac{1}{9})^{k-1}(\frac{2}{3}) = \frac{2}{3}\sum_{k=0}^{\infty}(\frac{1}{9})^k = \frac{2}{3} x \frac{1}{1-\frac{1}{9}} = \frac{2}{3} x \frac{9}{8} = \frac{3}{4}$

11) ('03 IB HL P2 #4)

(a) (i) Write down the integral whose value is E(X).

$$\int_0^2 \frac{1}{12}(8x - x^3)x\,dx$$

(ii) Hence evaluate E(X).

$$\int_0^2 \frac{1}{12}(8x - x^3)x\,dx = \frac{1}{12}\int_0^2 (8x^2 - x^4)dx = \frac{2}{9}x^3 - \frac{x^5}{60}\Big|_0^2 = \frac{56}{45}$$

(b) (i) Show that the median, m, of X satisfies the equation $m^4 - 16m^2 + 24 = 0$.

$$\int_0^n \frac{1}{12}(8x - x^3)dx = \frac{1}{2}$$

$$\frac{1}{12}(4x^2 - \frac{x^4}{4})\Big|_0^m = \frac{1}{2}$$

$$\frac{1}{3}m^2 - \frac{m^4}{48} = \frac{1}{2}$$

$m^4 - 16m^2 + 24 = 0$, *by multiplying through by 48 and rearranging terms.*

(ii) Hence evaluate m.

Using the quadratic, we get $m^2 = \dfrac{16 \pm \sqrt{256 - 4(24)}}{2} = 8 \pm \dfrac{\sqrt{160}}{2} = 8 \pm 2\sqrt{10}$

Of these two roots, only one, $8 - 2\sqrt{10}$, *could be the median, because we know That the median must be in between 0 and 2 inclusive. This value is approximately 1.675.*

(c) Evaluate the mode of X.

We must find the value of x for which the given function is maximum.

$f'(x) = \dfrac{1}{12}(8 - 3x^2)$. *Setting the derivative of the function, we find a turning point at* $x = \dfrac{2\sqrt{6}}{3} \approx 1.63$. *We can verify that this is a relative maximum because the derivative of f'(x) is a decreasing function over the interval [1,2].*

12) ('02 IB HL P2 #4)

(a) (i) Calculate the probability that Alan obtains a score of 9.

$\frac{4}{36}=\frac{1}{9}$ *, since the rolls (3,6), (4,5), (5,4) and (6,3) sum to 9 out of the 36 possible rolls.*

(ii) Calculate the probability that Alan and Belle both obtain a score of 9.

$$P(A=9\cap B=9)=\frac{1}{9}\times\frac{1}{9}=\frac{1}{81}$$, since their rolls are independent.

(b) (i) *This is tedious. In general, if the probability of obtaining some score is k, the probability both of them obtain it is k*^{2}*. So, we must sum over all 11 possible scores.*

$$P(A=B)=(\frac{1}{36})^2+(\frac{2}{36})^2+(\frac{3}{36})^2+(\frac{4}{36})^2+$$
$$(\frac{5}{36})^2+(\frac{6}{36})^2+(\frac{5}{36})^2+(\frac{4}{36})^2+$$
$$(\frac{3}{36})^2+(\frac{2}{36})^2+(\frac{1}{36})^2=\frac{146}{1296}=\frac{73}{648}$$

(ii) $P(A>B)=P(B>A)$, so $P(A>B)=\frac{1-P(A=B)}{2}=\frac{1-\frac{73}{648}}{2}=\frac{575}{1296}$

(c) (i) The probability of obtain a score of x or less is $\frac{x}{6}$, since there are x choices out of six sides that will satisfy the requirement. Since each of the four dice rolls are independent of one another, we can calculate the probability that none of them exceeds x to be the product $(\frac{x}{6})\,(\frac{x}{6})\,(\frac{x}{6})\,(\frac{x}{6})=(\frac{x}{6})^4$, as desired

(ii)

x	1	2	3	4	5	6
P(X = x)	1/1296	15/1296	65/1296	175/1296	369/1296	671/1296

(iii) $E(X)=\frac{1}{1296}+\frac{2\times15}{1296}+\frac{3\times65}{1296}+\frac{4\times175}{1296}+\frac{5\times369}{1296}+\frac{6\times671}{1296}=\frac{6797}{1296}\approx5.24$