Rule of Universal Specification

This is a fairly obvious rule, but one that is important:

If an open statement is true for all possible replacements in the designated universe, then that open statement is true for each specific individual member in that universe.

Symbolically speaking, we have:

If $\forall x\ p(x)$ is true, then we know that $p(a)$ is true, for each a in the universe for x.

Here is a simple example using this rule. Consider the following premises:

1) Each actor/actress on the TV show Friends is a millionaire.
2) Jennifer Aniston is an actress on the TV show Friends.

Therefore, Jennifer Aniston is a millionaire.

Symbolically, consider setting up these three statements:
p(x): x is an actor/actress on Friends.
q(x): x is a millionaire.

Now, the given information is $\forall x\ [p(x) \Rightarrow q(x)]$
If we wish to determine the financial status of Jennifer Aniston, we add into our premise the statement p(Jennifer Aniston) as being true.

Using the Rule of Universal Specification, and Rule of Detachment, we can conclude that q(Jennifer Aniston) is true; that is Jennifer Aniston is a millionaire.

Let's go ahead and look at another example in greater detail.

Consider each of these open statements for the next example:

p(x): x is a show on prime-time TV
q(x): x is a show on late-night TV
r(x): x is a soap opera

Now, consider the following argument:

No soap opera is on prime-time TV or late-night TV
All My Children is a soap opera.
Therefore, All My Children is not on prime-time TV.

(Note: Let A stand for "All My Children")

1) $\forall x\ [p(x) \lor q(x) \Rightarrow \neg r(x)]$	**Premise**
2) $r(A)$	**Premise**
3) $p(A) \lor q(A) \Rightarrow \neg r(A)$	**Step 1 & Rule of Univ. Spec.**
4) $\neg\neg r(A) \Rightarrow \neg(p(A) \lor q(A))$	**Contrapositive**
5) $r(A) \Rightarrow \neg(p(A) \lor q(A))$	**Law of Double Negation**
6) $r(A) \Rightarrow (\neg p(A) \land \neg q(A))$	**Step 5 & De Morgan's Law**
7) $\neg p(A) \land \neg q(A)$	**Steps 2 & 6 & Rule of Detachment (Modus Ponens)**
8) $\therefore \neg p(A)$	**Step 7 & Rule of Conjunctive Simplification**

The Rule of Universal Generalization

If an open statement p(x) is proved to be true when x is replaced by any arbitrarily chosen element c from our universe, then the universally qualified statement ∀x p(x) is true. (This rule also extends beyond one variable.)

We can use this to formally show that IF

∀x [p(x) ⇒ q(x)] AND
∀x [q(x) ⇒ r(x)] THEN
∀x [p(x) ⇒ r(x)].

1) ∀x [p(x) ⇒ q(x)]	Premise
2) p(c) ⇒ q(c)	Step 1 & Rule of Universal Specification
3) ∀x [q(x) ⇒ r(x)]	Premise
4) q(c) ⇒ r(c)	Step 3 & Rule of Universal Specification
5) p(c) ⇒ r(c)	Steps 2 & 4 and Law of Syllogism
6) ∴ ∀x [p(x) ⇒ r(x)]	Step 5 and the Rule of Universal Generalization

Now, consider the following assumptions
$\forall x \; p(x)$ **AND**
$\forall x \; [p(x) \wedge q(x) \Rightarrow r(x)]$

And use those to prove:

$\therefore \forall x \; [\neg r(x) \Rightarrow \neg q(x)].$

1) $\forall x \; p(x)$	**Premise**
2) $p(c)$	**Step 1 & Rule of Universal Specification**
3) $\forall x \; [p(x) \wedge q(x) \Rightarrow r(x)]$	**Premise**
4) $p(c) \wedge q(c) \Rightarrow r(c)$	**Step 3 & Rule of Universal Specification**
5) $T \wedge q(c) \Rightarrow r(c)$	**Substitution from Step 2**
6) $q(c) \Rightarrow r(c)$	**Identity Law**
7) $\neg r(c) \Rightarrow \neg q(c)$	**Contrapositive**
8) $\therefore \forall x \; [\neg r(x) \Rightarrow \neg q(x)]$	**Step 7 & Rule of Universal Generalization**

To make this example more concrete, consider the following open statements for p(x), q(x) and r(x)

Let the universe x be of all 4 sided polygons.

p(x): x is a quadrilateral.
q(x): x has four equal angles.
r(x): x is a rectangle.

Using basic geometry definitions, we find that

$\forall$**x p(x) (All 4 sided polygons are quadrilaterals.)**
$\forall$**x [p(x) $\land$ q(x) $\Rightarrow$ r(x)] (All 4 sided polygons that are**
                            quadrilaterals and have four equal
                            angles are rectangles.)
Therefore, all 4 sided polygons that are not rectangles do not have four equal angles, OR
$\therefore \forall$**x [$\neg$r(x) $\Rightarrow$ $\neg$q(x)]**

Examples illustrating proof techniques

We will use these definitions in the following problems:

An integer n is even if and only if there exists another integer r such that n = 2*r.

An integer n is odd if and only if there exists another integer r such that n = (2*r) + 1

If y | x, which is read as "x is divisible by y", or "y divides evenly into x", then x = yc, for some integer c. Remember in this definition, y must be non-zero.

In the book there are proofs showing that the sum of two odd numbers is even and that the product of two odd numbers is odd. I will show you a proof of slightly more interesting result.

<u>The square of an even number k is divisible by 4.</u>
Since we know k is even, we have k=2*r for some integer r.
Now, we can compute k^2.

$k^2 = (2*r)^2 = 4*r^2$, which is divisible by 4 since r^2 is an integer.

<u>Also, the square of an odd number k leaves a remainder of 1 when divided by 4.</u>
Since we know that k is odd, we have k=2*r+1 for some integer r. Now, we can compute k^2.

$k^2 = (2*r+1)^2 = 4*r^2+4*r+1 = 4(r^2+r) + 1$, which leaves a remainder of 1 when divided by 4 since 4 divides evenly into $4(r^2+r)$. (Because r^2+r must be an integer...)

In both of these examples, we used the Universal Rule of Generalization because we proved the result for arbitrary odd and even integers, and that implies that the general statement is true for all of them.

Prove that if n is an integer then n^2+n is an even integer

Proof #1: direct proof by method of exhaustion using cases.

All integers are even or odd. (I have not formally proved this, but you can use this fact throughout the class...) Thus, we have two different possibilities for n:

n is even: Then there is an integer r such that n=2r, then we have

$$n^2+n = (2r)^2 + 2r$$
$$= 4r^2 + 2r$$
$$= 2(2r^2 + r),$$

since this value is divisible by 2, it is even.

n is odd: Then there is an integer r such that n=2r+1, then we have

$$n^2+n = (2r+1)^2 + (2r+1)$$
$$= 4r^2 + 4r + 1 + 2r + 1$$
$$= 4r^2 + 6r + 2$$
$$= 2(2r^2 + 3r + 1),$$

and this value is also divisible by 2, thus it is even.

One way to analyze how and why this proof is sufficient is to break it down into a logical argument. Let p(n), q(n), and r(n) be the following open statements:

p(n): n is an even number.
q(n): n is an odd number.
r(n): n^2+n is even.

Since all integers are either even or odd, our goal is to prove

$(p(n) \vee q(n)) \Rightarrow r(n)$.

Logically, we can show this is equivalent to:

$(p(n) \Rightarrow r(n)) \wedge (q(n) \Rightarrow r(n))$

The first part of our proof showed that $(p(n) \Rightarrow r(n))$ is true, while the second part of our proof showed that $(q(n) \Rightarrow r(n))$ is true as well. Thus, together, these parts prove the original claim.

Proof #2: Contradiction

For contradiction's sake, assume that n^2+n is odd. We have the following: $n^2+n = n(n+1)$

We know that for the product of two integers to be odd, both of them have to be. (Part of this is shown in the book...)

However, it is impossible for both n and n+1 to be odd. (This is something you would have to prove also.)

Thus, the way contradiction works is that you assume that the result is false. Then use algebra and other rules to show that the premise must be false as well, or that something impossible occurs if the incorrect result is assumed.

The reason is that you came to an incorrect conclusion. You must have arrived at it by making an incorrect step. The ONLY possible incorrect step that could have been taken was the assumption made in the beginning. Hence, that is wrong, which implies the truth of what you are trying to prove.

In this class, you will often be given statements that may or may not be true. Your job will be to determine which is the case, and give proof of your answer. Typically, disproving an assertion is easier than proving one. Here are a couple examples illustrating how to disprove an assertion:

1) For all prime numbers greater than 100, the sum of their digits is greater than 4.

This is not true. We can verify that 101 is prime by showing that 2, 3, 5, and 7 do not divide into it evenly. Furthermore, we can see the sum of the digits in this number is 2, which is less than or equal to 4. Thus, by finding one prime number greater than 100 that has digits that sum to 2, we have disproved the claim.

Thus, to disprove a "for all" statement, all we had to do was find one value for which the statement did not hold.

Here is another example of method of exhaustion:

All integers in between 10 and 20 are either prime or have 2 or 3 as a factor.

Proof:
10 = 2*5
11 is prime
12 = 3*4
13 is prime
14 = 2*7
15 = 3*5
16 = 2*8
17 is prime
18 = 2*9
19 is prime
20 = 2*10

Here is an example where we prove a statement by proving its contrapositive:

If x + y is odd, then exactly one of x and y is odd.

The contrapositive of this statement is:

If either neither or both of x and y is odd, then x+y is even.

If we prove this statement, we have proven the original.

Thus, we can split the problem up into two cases:

Case 1:
Neither of x and y is odd, thus both are even.
Let x = 2a and y = 2b for some integers a and b.
Then,
x + y = 2a + 2b
    = 2(a+b).

At this point we can conclude that x+y is even since it is divisible by 2. (We know this because a+b must be an integer.)

Case 2:
Both of x and y are odd.
Let $x = 2a+1$, $y=2b+1$, for some integers a and b.
Then,
$$x+y = 2a+1+2b+1$$
$$= 2a+2b+2$$
$$= 2(a+b+1)$$
As before, we can conclude that x+y is even since it is a multiple of 2.