The Use of Quantifiers

Open Statement: One that contains a variable, and becomes a (single) statement when that variable is replaced with a value.

Example of an open statement:

7 divides evenly into x+7.

As you can see, for certain values of x, such as 7 or 21, this statement is true, but for other values, it is not true. (Note: It is possible for an open statement to always be true, such as "x is greater than x-1.")

We can denote the open statement above as p(x). Thus we can say that p(7) is true, whereas p(3) is not.

It is also possible for open statements to contain more than one variable. Consider the following:

x is a prime number that divides into y evenly AND is less than or equal to y.

We can denote the statement above as q(x,y). So, q(2,18) is true while q(5,13) and q(8,24) are both false.

With both of these open statements, we see that there are values for which the statements are true. Thus, it is reasonable to say something like the following:

For some x, p(x), and

for some x and y, q(x,y).

Quantifier Notation

Similarly, certain open statements can be true for all values of the variables involved. If we let r(x) be the statement: "x is greater than x-1," then we can make the claim:

For all x, r(x).

(Of course, whenever we say for all x, we must define what our universe, or possible x's are. Are they all the integers? All the real numbers? All the complex numbers? We will talk about how to specify that later.)

Since we are dealing with mathematics, we need short symbols to replace the English words "for some" and "for all". (A mathematician's whole goal in life is to confuse non-mathematicians :))

The symbol for the words "for some," which is more accurately describe as "there exists a value of," is $\exists$.

The symbol for the words "for all values of" is $\forall$.

So, we could make the following claims:

$\exists x [p(x)]$ and $\exists x \exists y [q(x)]$.

In English, these would be read as:

There exists a value of x for which 7 divides evenly into x+7.

There exists values for x and y for which x is a prime number that divides into y evenly AND is less than or equal to y.

As you can see, with just these two quantifiers we can put together even more complex statements. Many theorems, results, etc. in mathematics are typically represented using quantifiers. For example, we know that all positive integers have at least one prime factor that is less than or equal to the number itself. Mathematically speaking, we can use our statement q(x,y) from before to express this result.

q(x,y): x is a prime number that divides into y evenly AND is less than or equal to y.

$\forall y \exists x\ [\ q(x,y)\ ]$

Literally, this reads: "For all values y, there exists a value for x for which x is a prime number that divides into y evenly AND is less than or equal to y."

A very important distinction to make here is the order of the quantifiers. The statement

$\exists x \forall y\ [\ q(x,y)\ ]$ for example is NOT true.

This literally reads: "There exists a value of x for which for all values of y, x is a prime number that divides into y evenly AND is less than or equal to y."

The reason this is not true is because no matter what value you try to pick for x, you can ALWAYS find a value for y such that x DOES NOT divide y.

Perhaps, some practice evaluating these statements might help. Consider these simple open statements:

$p(x):\ x > 0$
$q(x):\ x^2 - 2x - 3 = 0$
$r(x):\ x < 0$
$s(x):\ x^2 > 0$

Decide whether each of these assertions is true or not:

1) $\exists x\ [p(x) \wedge q(x)\ ]$
2) $\forall x[p(x) \wedge r(x) \Rightarrow s(x)]$
3) $\forall x\ [r(x) \Rightarrow \neg q(x)]$

Now, take these assertions (which may not actually be true) and put them into symbolic form:

1) **For all values of x less than 0, x^2 is greater than 0.**

2) **There exists a value of x for which if $x^2 - 2x - 3 = 0$ then x is greater than 0.**

3) **For all values of x such that $x^2 - 2x - 3 = 0$, $x < 0$.**

A little chart to keep all of this straight
(Also on pg. 94 of the book...)

Statement	When True	When False
$\exists x\ p(x)$	For at least one value a in the universe for x, p(a) is true.	For all values a in the universe for x, p(a) is false.
$\forall x\ p(x)$	For all values a in the universe for x, p(a) is true.	For at least one value a in the universe for x, p(a) is false.
$\exists x\ [\neg p(x)]$	For at least one value a in the universe for x, p(a) is false.	For all values a in the universe for x, p(a) is true.
$\forall x\ [\neg p(x)]$	For all values a in the universe for x, p(a) is false.	For at least one value a in the universe for x, p(a) is true.

Given a statement ∀x [p(x) ⇒ q(x)], here are other related statements:

1) **Contrapositive:** ∀x [¬q(x) ⇒ ¬p(x)]
2) **Converse:** ∀x [q(x) ⇒ p(x)]
3) **Inverse:** ∀x [¬p(x) ⇒ ¬q(x)]

As we have mentioned before, if the given statement is true, then the contrapositive MUST BE true. However, neither the converse nor the inverse are necessarily true.

However, if both the given statement AND the converse are true, we have and if and only if relationship. Similarly, if both the given statement and the inverse are true, we also have and if and only if relationship.

A couple more rules that follow...

So now, consider each of these implications:

1) ∃x [p(x) ∧ q(x)] ⇒ [∃x p(x) ∧ ∃x q(x)]

2) ∃x [p(x) ∨ q(x)] ⇔ [∃x p(x) ∨ ∃x q(x)]

3) ∀x [p(x) ∧ q(x)] ⇔ [∀x p(x) ∧ ∀x q(x)]

4) [∀x p(x) ∨ ∀x q(x)] ⇒ ∀x [p(x) ∨ q(x)]

An important observation is that the first and fourth rules only go one way. Can you think of some counter examples to these rules?

Here are a couple of mine:

For number 1, let statement

p(x) = "x is greater than 100."
q(x) = "x is less than 0."

Certainly, ∃x p(x), also we have ∃x q(x). This is because there exists an x greater than 100, namely 101. But there also exists an x less than 0, namely –1.

BUT, it is NOT true that ∃x [p(x) ∧ q(x)]. If this were the case, then we could find a single value of x for which BOTH p(x) AND q(x) hold. But, clearly they are contradictory statements and this can not happen.

A counter example to the converse of rule number 4 is the following:

Let

p(x) = "x is an even integer."
q(x) = "x is an odd integer."

Clearly, for all integers x, we have either p(x) or q(x). BUT, it is not true that ∀x p(x) and it is also not true that ∀x q(x). The reason neither of these is true is because all integers are not odd, and all integers are not even either.

Other practice problems

First translate these assertions into English. Then deduce, with proof, whether they are valid or not. To show an assertion to be invalid, merely present a counter example. To prove it, you must show that the statement holds for all the values it says it will hold.

1. $\forall y \exists x \ [\, y = 2x \wedge x \in Z \wedge y \in Z \,]$

2. $\exists x \forall y \ [\, y/x < |y| \wedge x \in Z \wedge y \in Z \,]$

3. $\exists x \exists y \ [\, x/y = y/x \wedge x \in Z \wedge y \in Z \,]$