

Fundamentals of Logic

Statements/Propositions – Sentences that are true or false but not both. (Just like a simple boolean expression or conditional expression in a programming language.)

For our purposes, statements will simply be denoted by lowercase letters of the alphabet, typically p , q , and r . I will refer to these as boolean variables as well.

Given simple statements, we can construct more complex statements using logical connectives. Here are 4 logical connectives we will use:

- 1) **Conjunction:** This is denoted by the ' \wedge ' symbol. The statement $p \wedge q$ is read as “ p and q .” Only if both the values of p and q are true does this expression evaluate to true. Otherwise it is false.
- 2) **Disjunction:** This is denoted by the ' \vee ' symbol. The $p \vee q$ is read as “ p or q .” As long as at least one of the values of p or q is true, the entire expression is true
- 3) **Implication:** This is denoted by the ' \Rightarrow ' symbol. The statement $p \Rightarrow q$ is read as “ p implies q ”. Essentially, in a programming language, this logic is captured in an if then statement. If p is true, the q must be true. However, if p is not true, there is no guarantee of the truth of q . An important observation to note: when statements are combined with an implication, there is no need for there to be a causal relationship between the two for the implication to be true.

Consider the following implications:

If my bread is green, then I will not eat it.

Here, if the bread is not green, that does not guarantee that I will eat it. Perhaps it is wheat bread and I hate wheat bread. All I know is that if my bread is green, I will definitely NOT eat it.

If Pluto is the largest planet in our solar system, then pigs will fly out of my butt.

Wayne would actually be making a correct implication here, (assuming that we currently have accurate knowledge about our solar system...) Since the first part of our implication is false, the entire implication is automatically true.

4) Biconditional: This is denoted by the ' \Leftrightarrow ' symbol, and the statement $p \Leftrightarrow q$ is read "p if and only if q." The phrase "if and only if" is often abbreviated as "iff". Breaking this down into pieces, this means that $p \Rightarrow q$ AND $q \Leftarrow p$. Hence exactly when p is true, q is true. (And in all cases where p is false, q must be false as well.)

Here is an example of a biconditional:

If and only if my alarm clock rings in the morning, then I will attend my morning classes.

From this statement we CAN deduce that if the alarm clock does NOT ring, then I will NOT go to my morning classes.

Finally, it will be important to have a symbol to denote the negation of a statement/proposition. We will use the ‘ \neg ’ symbol. The statement $\neg p$ will be read “not p.” This statement will have the opposite value of p.

Truth Tables

Now, given a particular compound statement, we can use a truth table to determine which values of the boolean variables result in the statement being true.

The idea here is to simply make a table, listing all the possible combinations of values for each of the boolean variables in a statement. Then, plug these values into the statement and see if it is true or not with these values. This is probably easiest seen with an example.

Consider the statement: $p \wedge (q \vee \neg r)$. Here is a truth table:

p	q	r	$q \vee \neg r$	$p \wedge (q \vee \neg r)$
0	0	0	1	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	0
1	0	0	1	1
1	0	1	0	0
1	1	0	1	1
1	1	1	1	1

Thus there are three possible combinations of values for p, q, and r that make $p \wedge (q \vee \neg r)$ true.

To see an example, let p, q and r be the following statements:

p: I have taken out the trash.

q: I have finished cleaning the dishes.

r: I have watched 5 hours of TV.

Assume that a child is a good one if $p \wedge (q \vee \neg r)$ holds true, and bad otherwise. Under what conditions is a child good?

If it turns out that a statement is always true (such as $p \vee \neg p$), then it is called a *tautology*.

If a statement is always false (such as $p \wedge \neg p$), then it is called a contradiction.

EXERCISE: Make a truth table for the statement: $(p \wedge \neg q) \Rightarrow r$.

Algebra of Propositions

It would be nice if we had some methodology for determining if two logical expressions are equal, or a method of simplifying a given logical expression.

Perhaps the most obvious way to check for the equality of two logical expressions it to write out truth tables for both. However, this could be quite tedious. (But it does always work...)

Two statements s and t are considered to be logically equivalent if $s \Leftrightarrow t$.

We will need some laws to simplify logical expressions. Here is the list of laws from the text:

- 1) $\neg\neg p \Leftrightarrow p$ (Law of Double Negation)
- 2) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ (De Morgan's Laws)
- 3) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- 4) $p \vee q \Leftrightarrow q \vee p$ (Commutative Laws)
- 5) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ (Associative Laws)
- 6) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ (Distributive Laws)
- 7) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- 6) $p \vee p \Leftrightarrow p$ (Idempotent Laws)
- 7) $p \vee F \Leftrightarrow p, p \wedge T \Leftrightarrow p$ (Identity Laws)
- 8) $p \vee \neg p \Leftrightarrow T, p \wedge \neg p \Leftrightarrow F$ (Inverse Laws)
- 9) $p \vee T \Leftrightarrow T, p \wedge F \Leftrightarrow F$ (Domination Laws)
- 10) $p \vee (p \wedge q) \Leftrightarrow p$ (Absorption Laws)
- 11) $p \wedge (p \vee q) \Leftrightarrow p$

We can use these laws to simplify a statement. Consider the following statement:

$$\begin{aligned} (p \vee q) \wedge \neg(\neg p \wedge q) &\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q) \quad \text{(DeMorgans)} \\ &\Leftrightarrow (p \vee q) \wedge (p \vee \neg q) \quad \text{(Double Neg.)} \\ &\Leftrightarrow p \vee (q \wedge \neg q) \quad \text{(Dist. Law)} \\ &\Leftrightarrow p \vee F \quad \text{(Inverse)} \\ &\Leftrightarrow p \quad \text{(Identity)} \end{aligned}$$

Something else that will be helpful to us will be to expression implications using just and's and or's. By writing out truth tables, we find that

$$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q) \quad \text{(We'll call this the implication identity.)}$$

Using this, we can also come up for an expression without implications that is equivalent to $p \Leftrightarrow q$.

As an exercise, simplify the following expression, giving the reason for each simplification.

$$(p \Rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$$

Methods to Prove Logical Implications

The standard form of an argument (or theorems) can be represented using the logical symbols we have learned so far:

$$(p_1 \wedge p_2 \wedge p_3 \wedge \dots p_n) \Rightarrow q$$

Essentially, to show that this statement is always true (a tautology), we must show that if all of p_1 through p_n are true, then q must be true as well.

Another way to look at this is that we must show that if q is ever false, then at least ONE of p_1 through p_n must be false as well. (This is the contrapositive of the original assertion.)

Let's look at an example of how you might go about proving a statement in a general form, given some extra information.

Let p , q , and r be the following statements:

p : Daunte Culpepper throws for 4000 or more yards in 2006.

q : Daunte Culpepper gets injured in 2006.

r : The Dolphins win the Super Bowl for the 2006 season.

Let the premises be the following:

p_1 : If Daunte Culpepper does NOT get injured in 2006, he will throw for 4000 yards or more.

p_2 : If Daunte Culpepper throws for 4000 or more yards in 2006, the Dolphins will win the Super Bowl for the 2006 season.

p_3 : Daunte Culpepper did NOT get injured in 2006.

**Note: I realize this whole exercise is 100% hypothetical and not based upon any actual game play from the real current NFL season.*

Now, I want to show that

$$(p_1 \wedge p_2 \wedge p_3) \Rightarrow r$$

First, I must express p_1 , p_2 , and p_3 in terms of p , q , and r .

$$p_1 : \neg q \Rightarrow p$$

$$p_2 : p \Rightarrow r$$

$$p_3 : \neg q$$

Thus, we are trying to show the following:

$$[(\neg q \Rightarrow p) \wedge (p \Rightarrow r) \wedge (\neg q)] \Rightarrow r$$

We can prove that this is a tautology by using a truth table and verifying that this expression is true for all 8 possible sets of values for p , q and r .

Another way we can show this statement is to use the laws of logic to simplify the statement as follows:

$$[(\neg\neg q \vee p) \wedge (\neg p \vee r) \wedge (\neg q)] \Rightarrow r \quad \text{(Identity for implication)}$$

$$[(q \vee p) \wedge (\neg p \vee r) \wedge (\neg q)] \Rightarrow r \quad \text{(Double Negation)}$$

$$\neg [((q \vee p) \wedge (\neg p \vee r)) \wedge (\neg q)] \vee r \quad \text{(Identity for implication)}$$

$$\neg((q \vee p) \wedge (\neg p \vee r)) \vee \neg(\neg q) \vee r \quad \text{(De Morgan's Law)}$$

$$\neg((q \vee p) \wedge (\neg p \vee r)) \vee q \vee r \quad \text{(Double Negation)}$$

$$\neg(q \vee p) \vee \neg(\neg p \vee r) \vee q \vee r \quad \text{(De Morgan's Law)}$$

$$(\neg q \wedge \neg p) \vee (\neg\neg p \wedge \neg r) \vee q \vee r \quad \text{(De Morgan's Law)}$$

$$(\neg q \wedge \neg p) \vee (p \wedge \neg r) \vee q \vee r \quad \text{(Double Negation)}$$

$$(r \vee (p \wedge \neg r)) \vee (q \vee (\neg q \wedge \neg p)) \quad \text{(Associate+Commutative)}$$

$$((r \vee p) \wedge (r \vee \neg r)) \vee ((q \vee \neg q) \wedge (q \vee \neg p)) \quad \text{(Distributive)}$$

$$((r \vee p) \wedge T) \vee (T \wedge (q \vee \neg p)) \quad \text{(Inverse Laws)}$$

$$(r \vee p) \vee (q \vee \neg p) \quad \text{(Identity Laws)}$$

$$(\neg p \vee p) \vee (r \vee q) \quad \text{(Associate+Commutative)}$$

$$T \vee (r \vee q) \quad \text{(Inverse Laws)}$$

$$T \quad \text{(Domination Laws)}$$

When $p \Rightarrow q$ is a tautology, we say that the statement is a logical implication. This implication is equivalent to the following:

$\neg q \Rightarrow \neg p$, which is the contrapositive as mentioned before. In certain situations, it will be easier to prove the contrapositive than the original statement.

Getting back to the example we just went over, we have shown that any set of statements p_1 , p_2 , and p_3 of the form above imply the statement r .

Now, you might notice that proving the example above by either using the truth table method or using the laws of logic took a great deal of effort for a relatively intuitive result. (Just by hearing the premises, you probably already knew that statement r followed logically.)

If we examine a truth table, we will find that in most cases (most of the rows), at least one of the premises is false anyway. If this is the case, there is no need to even compute the value of the conclusion. So, we have an indication that we could trim some work.

Also, it seems logical to have some rules of inference, rather than having to turn each inference into an equivalent logical expression without an inference. (What we will do is verify these rules with truth tables. Once they are proved, then we can use them on their own.)

Next lecture, I will derive three of these rules, then show you the rest of them.