COT 3100 Practice Induction Problems for Lecture

1) Prove, using mathematical induction, that for all positive integers n, we have

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \log(n) + 1$$

where $\log(n)$ denotes the natural logarithm. You may use that $\log(n+1) - \log(n) \geq \frac{1}{n+1}$ holds for all positive integers n.

<u>Proof</u>

Base case: n = 1. The left-hand side of this inequality evaluated at n = 1 is 1.
The right hand side of this inequality evaluated at n = 1 is log(1) + 1 = 1.
Thus, the inequality holds for n = 1.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \leq \log(k) + 1$$

Inductive step: Prove for n = k + 1 that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} \leq \log(k+1) + 1$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\mathrm{k}}\right) + \frac{1}{\mathrm{k}+1}$$

$$\leq \log(k) + 1 + \frac{1}{\mathrm{k}+1}, \text{ using I. H.}$$

$$= \left(\log(k) + \frac{1}{k+1}\right) + 1$$

$$\leq \log(k+1) + 1, \text{ since using the inequality given it follows that}$$

$$\log(k+1) \geq \log(k) + \frac{1}{k+1}$$

This completes proving the inductive step. It follows that the original assertion holds for all positive integers n.

2) Prove, using mathematical induction, that for all non-negative integers n, $10 \mid (9^{n+1} + 13^{2n})$.

Proof

Base case: n = 0. Plug into the expression to get $9^{0+1} + 13^{2(0)} = 9 + 1 = 10$. Since $10 \mid 10$, the base case holds.

Inductive Hypothesis (IH): Assume for an arbitrary non-negative integer n = k, that $10 \mid (9^{k+1} + 13^{2k})$. Equivalently, there is some integer c such that $10c = 9^{k+1} + 13^{2k}$.

Inductive Step: Prove for n = k+1 that $10 \mid (9^{k+1+1} + 13^{2(k+1)})$.

$$
\begin{aligned}
9^{k+1+1} + 13^{2(k+1)} &= 9^{k+2} + 13^{2k+2} \\
&= 9^1 9^{k+1} + 13^2 13^{2k}, \text{ since } a^{b+c} = a^b a^c. \\
&= 9(9^{k+1}) + 169(13^{2k}) \\
&= 9(9^{k+1}) + (160 + 9)(13^{2k}) \\
&= 9(9^{k+1}) + 9(13^{2k}) + 160(13^{2k}) \\
&= 9(9^{k+1} + 13^{2k}) + 160(13^{2k}) \\
&= 9(10c) + 160(13^{2k}), \text{ using the integer c defined in the IH} \\
&= 10(9c + 16(13^{2k}))
\end{aligned}
$$

Since 9c, 16 and 13^{2k} are all integers, it follows that the expression above is divisible by 10, proving the inductive hypothesis.

3) Using mathematical induction, prove the following statement P(n):

$$\text{For all integers } n > 1,\ \sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$$

Proof

(1) Base case: n =2, check P(2) is true.

LHS = $\sum_{i=1}^{2} \frac{1}{i^2} = 1 + \frac{1}{4} = \frac{5}{4}$, RHS = $2 - \frac{1}{2} = \frac{3}{2} = \frac{6}{4}$.

LHS < RHS, so the base case holds (or P(2) is true).

(2) Inductive hypothesis: Assume for an arbitrary positive integer n = k (k > 1) that

$$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}.\ \text{(or Assume P(k) is true)}$$

(3) Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k+1}$

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} &= \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \\
&< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \qquad \text{Using the induction hypothesis} \\
&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2} \right) \\
&= 2 - \left(\frac{k^2 + k + 1}{k(k+1)^2} \right) \\
&= 2 - \frac{k(k+1)}{k(k+1)^2} - \frac{1}{k(k+1)^2} \\
&= 2 - \frac{1}{(k+1)} - \frac{1}{k(k+1)^2} \\
&< 2 - \frac{1}{(k+1)} \ \text{ as desired.}
\end{aligned}$$

Based on the logic of mathematical induction, this proves that the given assertion is true for all integers n>1.

4) Let T(n) be a recurrence relation defined by: $T(1) = 2$,

$T(n) = 2nT(n-1)$, for $n > 1$.

Prove that for all positive integers, $T(n) = 2^n n!$.

<u>Proof</u>

Base Case: n = 1. LHS = $T(1) = 2$, RHS = $2^1 1! = 2$. Both sides are equal, satisfying the base case.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that $T(k) = 2^k k!$.

Inductive Step: Prove for n = k+1 that $T(k+1) = 2^{k+1}(k+1)!$.

$T(k+1) = 2(k+1)T(k)$
$= 2(k+1)2^k k!$, plugging in the IH
$= 2^{k+1}(k+1)k!$
$= 2^{k+1}(k+1)!$

This proves the inductive step and completes the proof.

5) Whenever Binary Billy acts up, his punishment is to write binary numbers on the board. He always starts writing 0, 1, 10, 11, 100, etc. Depending on the severity of behavior, Billy has to write all the binary numbers starting at 0 upto all binary numbers with a certain number of digits. For example, if Billy's bad behavior was rated at a 5, then Billy would have to write all the binary numbers from 0 through 11111. Let B(n) denote the total number of binary *digits* Billy must write for a bad behavior rating of n. Using induction on n, prove that $B(n) = (n-1)2^n + 2$, for all positive integers n.

Proof

Base case. n=1. Billy must write two digits, 0 and 1, thus B(1) = 2. Looking at the right hand side we find it equal to $(1-1)2^1 + 2 = 2$, thus the given formula is true for n=1.

Inductive hypothesis: Assume that for an arbitrary value of n=k that $B(k) = (k-1)2^k + 2$.

Inductive step: Prove, under the inductive hypothesis, that for n=k+1, $B(k+1) = ((k+1)-1)2^{k+1} + 2 = k2^{k+1} + 2$.

Let b(n) = number of binary numbers with EXACTLY n digits. For example, b(2) = 2 since there are two binary numbers, 10 and 11 with exactly 2 digits.

In particular, using the multiplication principle, we have that $b(n) = 2^{n-1}$. This is because the first binary digit of the n digits must be 1. The rest can either be 0 or 1. Thus, we have 2 choices for each of the n-1 remaining digits. Since each of these choices are independent, we multiply 2 by itself n-1 times to obtain the total number of possible binary numbers with exactly n digits.

Now, to prove the assertion:

$B(k+1) = B(k) + (k+1)*b(k+1)$, since we are adding b(k+1) numbers with k+1 digits

$$\begin{aligned}
&= (k-1)2^k + 2 + (k+1)2^{(k+1)-1} \\
&= (k-1)2^k + 2 + (k+1)2^k \\
&= ((k-1)+(k+1))2^k + 2 \\
&= (2k)2^k + 2 \\
&= k2^{k+1} + 2
\end{aligned}$$

6) Prove for all positive integers n, $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{bmatrix}$.

<u>Proof</u>

Base case: n =1 LHS = $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, RHS = $\begin{bmatrix} 3^1 & 3^1 - 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$.
The two sides are equal so the base case holds.

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

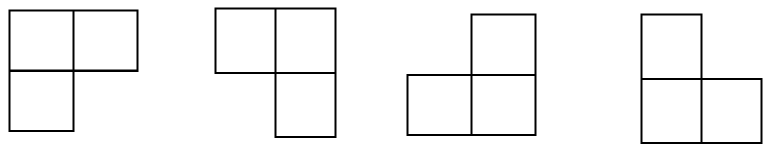
$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{bmatrix}.$$

Inductive step: Prove for n = k+1 that $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{bmatrix}$.

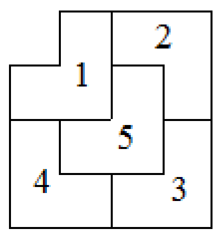
$$\begin{aligned}
\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} &= \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^k \\
&= \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3(3^k) + 2(0) & 3(3^k - 1) + 2(1) \\ 0(3^k) + 1(0) & 0(3^k - 1) + 1(1) \end{bmatrix} \\
&= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 3 + 2 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{bmatrix}, \text{ as desired.}
\end{aligned}$$

This proves that the given assertion is true for all positive integers n.

7) A tromino is a tile consisting of three unit squares in an L shape. The following are the four possible orientations a tromino can be placed:



Using induction on n, prove that for all non-negative integers, n, a 2^n x 2^n grid of unit squares with a single unit square removed can be tiled properly with a set of trominos. A proper tiling covers every unit square of the original object with a single unit square of a single tromino. For example, the following is a valid tiling of the 4 x 4 grid with the top left corner missing:

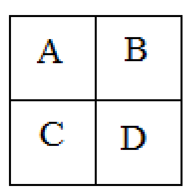


<u>Proof</u>
Base case: n=0, in this case we have a 2^0 x 2^0 square with one square missing, which means that what remains to be tiled is nothing, since we have one square missing from one square. Trivially, we can tile nothing with 0 L tiles.

Inductive hypothesis: For an arbitrary non-negative integer n = k, we can tile a 2^k x 2^k grid of unit squares with one square missing.

Inductive step: Prove for n = k+1 that we can tile a 2^{k+1} x 2^{k+1} grid of unit squares with one square missing.

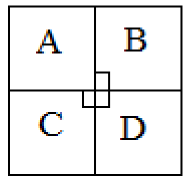
Given our square to tile, we partition it into 4 equal quadrants:



Note that since each quadrant is equal in size, each quadrant MUST BE a 2^k x 2^k square, since $2^k + 2^k = 2^{k+1}$. The missing unit square (that shouldn't be tiled) must be located in one of the four quadrants.

According to our inductive hypothesis, we can tile this quadrant (since it has a missing unit square).

Now, we have three quadrants left to tile. It must be the case that these three quadrants are "next to each other." For example, consider the case that the three quadrants are B, C and D. If this is the case, then we can place a single tromino at the center of the diagram above, with the tromino covering one unit square in B, one unit square in C and one unit square in D:



Now, we have tiled all of A with its square missing, as well as one unit square in quadrants B, C and D. Finally, we are left to tile quadrants B, C and D with ***exactly one unit square missing!!!*** We can perform this tiling based on the inductive hypothesis, completing the tiling of our original 2^{k+1} x 2^{k+1} design with a single unit square missing, completing the proof.