Problems involving functions

Consider a set $A = \{1, 2, 3\}$, a set $B = \{a, b, c\}$, and a set $C = \{x, y\}$. Define a relation $R \subseteq A \times B$ as $R = \{(1, a), (2, b), (2, c)\}$, a relation $S \subseteq B \times C$ as $S = \{(a, x), (b, y)\}$, and a relation $T \subseteq A \times B$ as $T = \{(1, a), (2, c), (3, b)\}$.

Is R a function? If so, is it an injection?

No, since 2 maps to two different elements in the set B.

Is R^{-1} a function? If so, is it an injection?

Yes, since each of a, b, and c in R^{-1} map to exactly one element. However, R^{-1} is not injective because both b and c map to the same element.

Is S a function? Is S^{-1} a function?

S is not a function because c does not map to any element in the set C. But, S^{-1} is a function since both x and y are mapped to exactly one element in the set B.

Is T a function? If so, is it a surjection?

T is a function. It maps each element in the set A to an unique element is the set B. Since both sets are of an equal size, T is a bijection (and a surjection, of course.)

Is the composed relation $R^{-1} \circ S^{-1}$ a function? If so, is it an injection? Is it a surjection?

The composition, $R^{-1} \circ S^{-1} = \{(x,1),(y,2)\}$. Thus it is a function since each of x and y map to an element in A. Let $f: C \rightarrow A$ be the function $R^{-1} \circ S^{-1}$. Then we have $f(x) \neq f(y)$

Is the composed relation $S \circ T$ a function? If so, is it an injection?

No, since 2 is not mapped to any element in the set C.

Define $h: R - \{2\} \rightarrow R$ where $h(x) = (x - 3) / (x - 2)$. Prove that $h(x)$ is an injection, but not a surjection.

We must show that if $h(a)=h(b)$, then $a=b$.

$h(a) = (a-3)/(a-2)$
$h(b) = (b-3)/(b-2)$

$(a-3)/(a-2) = (b-3)/(b-2)$
$((a-2)-1)/(a-2) = ((b-2) - 1)/(b-2)$
$1 - 1/(a-2) = 1 - 1/(b-2)$
$1/(a-2) = 1/(b-2)$, since $a,b\neq 2$, we can cross multiply.
$a - 2 = b - 2$
$a = b$.

h is NOT a surjection because there is no value a such that $h(a) = 1$. To see this note that $h(a) = 1 - 1/(a-2)$. Since $1/(a-2)$ can not equal 0, we find that no value of a makes $h(a)=1$.

Let f: $A \rightarrow B$ and g: $B \rightarrow C$ denote two functions. If a function $g°f$: $A \rightarrow C$ is a surjection, and g is an injection, prove that f is a surjection.

Since $g°f$ is a surjection, we must have that there exists an element x in A such that $g(f(x))=c$, for all elements c in the set C.

Furthermore, if g in an injection, we know that if $g(a)=g(b)$, then $a=b$.

We must now show that there exists an element y in A such that $f(y)=b$ for all elements b of the set B.

Assume to the contrary that there is an element b such that there is no y where $f(y)=b$. But, we know that there does exist an element x such that $g(f(x)) = c$ for all elements c of the set C.

Let $g(b) = c'$. Since g is a function from B to C, c' must be an element of C. But we also know that $g(f(x))=c'$, since $g°f$ is a surjection. Since we have assumed that there exists no element y in the set A such that $f(y) = b$, we must have that $b \neq f(x)$.

BUT, that gives us a situation were $g(b) = g(f(x))$, but $b \neq f(x)$, which contradicts the assumption that g was injective. Thus, our assumption must have been wrong meaning that f is injective.

Let $Z = \{0, 1, -1, 2, -2, \ldots\}$ denote the set of all integers (zero, positive, and negative). Define a function g: $Z \to Z$ by the following formula:

$$g(m) = \begin{cases} 1 - m, \text{ if } m \text{ is an even integer; otherwise,} \\ m + 3, \text{ if } m \text{ is an odd integer.} \end{cases}$$

(Thus, for example, $g(0) = 1 - 0 = 1$; $g(1) = 1 + 3 = 4$; etc.)

Prove that the function g defines a bijection from Z to Z; that is, prove that g is an injection (one-to-one) and g is a surjection (onto).

First, let's show that the function is an injection. We must show that if f(a) = f(b), then a = b.

Assume f(a) = f(b). We have four distinct cases:

Case 1: a and b are both even
Case 2: a and b are both odd
Case 3: a is even and b is odd
Case 4: a is odd and b is even

Case 1: f(a) = f(b) (Given)
1 – a = 1 – b (since a and b are even)
a = b

Case 2: f(a) = f(b) (Given)
a + 3 = b + 3 (since a and b are odd)
a = b

Case 3: f(a) = f(b)
1 – a = b + 3 (since a is even and b is odd)
a + b = -2
But, this is impossible since a and b are of different parity. So this case would never occur.

Case 4: $f(a) = f(b)$

$a + 3 = 1 – b$ (since a is odd and b is even)

$a + b = -2$

But, this is impossible since a and b are of different parity. So this case would never occur.

In all possible cases, we have shown that if $f(a)=f(b)$, then $a=b$.

Now we must show that the function is surjective. To do this, we can find the inverse function:

We have $g(m) = 1 – m$. Solving for m we have:
$m = 1 – g(m)$. (Where m is even and $g(m)$ is odd) So we have:
$g^{-1}(m) = 1 – m$, when m is odd, and $g^{-1}(m)$ is even.

$g(m) = m + 3$. Solving for m we have:
$m = g(m) – 3$. (Where m is odd and $g(m)$ is even)So we have:
$g^{-1}(m) = m – 3$, when m is even, and $g^{-1}(m)$ is odd.

The reason we "switched" the "when m is even/odd" is because whenever we put an odd input into the function the output was even and vice versa.

Now, we can show that the function is surjective:

Given any integer b, here is how to find an integer a such that $f(a)=b$:

If b is odd, $f(1 – b) = b$. (So here $a = 1 – b$.)
$f(1 – b) = 1 – (1 – b)$, since $1 – b$ is even...
$= b$
If b is even, $f(b – 3) = b$. (And here $a = b – 3$.)
$f(b – 3) = (b – 3) + 3$, since $b – 3$ is odd...
$= b$.

Give a small example of functions f and g such that f is an injection (one-to-one) and g is a surjection (onto), but $g \circ f$ is not an injection (one-to-one).

Let the set A = {1,2}, B={a,b,c}, C={x,y}

f(1) = a f(2) = c
g(a) = x g(b) = y g(c) = x
g(f(1)) = g(f(2)) = x

Suppose $f : A \to B$ and $g : B \to A$ are two functions. Assume that for any element $y \in B$, $(f \circ g)(y) = y$, but there is at least one element $x \in A$ such that $(g \circ f)(x) \neq x$. Prove that f is onto, but not one to one.

To prove that f is onto it is sufficient to show that for any $y \in B$ there exists some $x \in A$, such that $f(x) = y$. Take arbitrary $y \in B$. Since $g : B \to A$ is a function, we can find $x = g(y) \in A$. Since $f : A \to B$ is a function with domain A, we can find the image of x under f, $f(x) = f(g(y)) = (f \circ g)(y)$ by the definition of the function composition. We are given that $(f \circ g)(y) = y$. This means that we can always find $x = g(y) \in A$ such that $f(x) = y$, i.e. f is onto.

To prove that f is not one to one, it is sufficient to find two distinct elements of A with the same image under f. We are given that there exists $x \in A$ such that $(g \circ f)(x) \neq x$.
Let $(g \circ f)(x) = z \neq x$ and $y = f(x)$. Since we know that $f(g(y)) = y$ for any $y \in B$, we can imply that $f(z) = f(g(y)) = y$. So, we found $x \neq z$ such that $f(x) = f(z)$, i.e. the function f is not one-to-one.