Functions

A function is a special type of relation. In particular, here are the rules for a relation to be a function, for an arbitrary relation $R \subseteq A \times B$:

For each element in a $\in$ A, we must have EXACTLY 1 element in R such that a is the first term of the ordered pair. In English that means for each element in the set A it MUST BE related to exactly one element in the set B. Typically, we call the set A the domain and the set B the co-domain.

It is possible that all of the possible values of f(a) (when a $\in$ A) form only a proper subset of B. Thus the set of possible values of the function, which can be more formally written as follows:

$f(A) = \{ f(a) \mid a \in A \}$ is known as the range of the function.

Here is an example of a function :

Let the set A = { the set of words in the English language }
For all a $\in$ A, define f(a) as follows:

f(a) = the number of letters in the word a.

Thus, we have that the domain is the set of all English words, the co-domain is the set of all positive integers, and the range is all positive integers less than 29, (assuming that antidisestablishmentarianism is the longest word in the English language).

However, recall the relation I showed you in an earlier lecture:

Cocktails = {(Orange Juice, Vodka), (Cranberry Juice, Vodka), (Coke, Rum), (Orange Juice, Peach Schnapps) }

This is not a function because Orange Juice is related to more than one element of the range.

Note that you can define functions on multiple sets, just as you can define relations on multiple sets. Consider the following: f(a,b) = a+b.

This is a function of two real number variables. Notice that different ordered pairs of this function map to the same element in the range. For example, f(2,3) = f(4,1).

We have already talked about composing relations together. Now we will introduce function composition.

Consider the following example, which uses the first function from this lecture:

f(a) = the number of letters in the word a.
g(b) = the sum of the digits in the integer b.

Consider inputting the output from f(a) into the function g. Pictorially, we get something like this:

Composing Functions produces a function

Let f: A → B and g: B → C be two functions. The composition of f and g as relations defines a function g ∘ f : A → C, such that g ∘ f (a) = g(f(a)).

We need to show that for each element a ∈ A, the expression (or formula) g(f(a)) assigns the unique element of C related to a based on the composition of f and g considered as relations. By the definition of relation composition,

g ∘ f = {(a, c) | a ∈ A ∧ c ∈ C ∧ (∃b ∈ B | (a, b) ∈ f ∧ (b, c) ∈ g)}.

However, (a, b) ∈ f means b = f (a) using the function notation, and (b, c) ∈ g means c = g(b) by the function notation. Thus, the relation g ∘ f contains pairs (a, c) with a ∈ A and c ∈ C, such that c = g(b) = g(f(a)), by substitutions. Further, this element c that is related to element a via the composed relation g ∘ f is unique; thus, the relation g ∘ f defines a function g ∘ f : A → C.

Consider an example from math class. Let $f(x) = x^3$ and $g(x) = 2x - 1$. We have the following function compositions :

$f \circ g(x) = f(g(x)) = f(2x - 1) = (2x - 1)^3$
$g \circ f(x) = g(f(x)) = g(x^3) = 2x^3 - 1$

As can be seen from this example, typically, $f(g(x)) \neq g(f(x))$. Also, notice that the order in which function composition is written is OPPOSITE to the order in which relation composition is written. This is because of the standard mathematical notion of a function and it's traditional use.

Injection, Surjection and Bijection

These definitions are often confusing, but I think some pictures will help...

A function f is injective, or is one-to-one, if for all a, b ∈ A, a ≠ b ⟹ f(a) ≠ f(b). Equivalently, this means if f(a) = f(b) then a = b. We also say f is an injection in this case.

f is surjective, or is onto, if f(A) = B. That is, if for every b ∈ B there exists a ∈ A such that f(a) = b. We also say f is a surjection in this case.

f is bijective if it is both injective and surjective. We also say f is a bijection in this case.

Consider these questions:

Is the function $f(x) = x^2$ injective, where the domain and co-domain are all real numbers?

Is the function f(x) = sin x surjective, where the co-domain is the set of real numbers in the interval [-1,1].

Is the function $f(x) = x^3$ a bijection?

Consider functions $f: A \rightarrow B$ and $g: B \rightarrow C$, and the composition $g \circ f : A \rightarrow C$.

(a) If both f and g are injective, then g ∘ f is injective.
(b) If both f and g are surjective, then g ∘ f is surjective.
(c) If both f and g are bijective, then g ∘ f is bijective.

To prove g ∘ f is injective, we need to prove that for a, b ∈ A,

if g ∘ f (a) = g ∘ f (b) then a = b, by the definition of injection.

g(f(a)) = g(f(b))

Since g is injective, that means for these two to be equal, we must have f(a) = f(b).

However, we also know that f is injective, thus, this implies that a=b.

To prove g ∘ f is surjective, we need to prove that for all elements c ∈ C, there exists an element a such that g ∘ f (a) = c.

Let c be an arbitrary element from the set C.
Since g is surjective, we know there exists an element b such that g(b) = c.
Similarly, for this element b, since f is surjective, we know there exists an element a such that f(a) = b.
Substituting, we have g(f(a)) = c.
Thus, we have shown that for an arbitrarily chosen element c, we can always find an element a such that g ∘ f (a) = c.

Inverse Functions

A bijection f: A → B maps each element of A to a unique element of B, and conversely, each element of B has a unique element of A as its pre-image. The following theorem makes a precise statement of this relationship.

Let f: A → B be a bijection. Then the inverse relation of f, defined from B to A as {(b, a) | b ∈ B and a ∈ A, and f (a) = b}, is a function from B to A such that g ∘ f(a) = a for all a ∈ A, and f ∘ g(b) = b for all b ∈ B. The function g is called the inverse function of f, denoted f^{-1}.

Note that since f: A → B is a bijection, for each b ∈ B there exists one and only one element a ∈ A, such that f (a) = b. Thus, the relation from B to A, relating b ∈ B to its pre-image a ∈ A under f, is a function by the definition of function. That is, we have defined a function

g: B → A such that for b ∈ B, g(b) = a iff f (a) = b

g ∘ f (a) = g(f(a)),
      = g(b)
      = a.

f ∘ g(b) = f(g(b))
      = f(a)
      = b.

Let f: A → B and g: B → A be two functions. If g ∘ f(a) = a for all a ∈ A, and f ∘ g(b) = b for all b ∈ B, then both f and g are bijections, and they are inverse functions of each other, that is, g = f^{-1} and f = g^{-1}.

First, we will show that if g ∘ f(a) = a, for all a ∈ A and f ∘ g(b) = b, for all b ∈ B, then f is a bijection and g = f^{-1}

We first prove that f is an injection. That is, suppose f(a) = f(a'), for a, a' ∈ A, we want to prove that a = a'.

Using function g,
 g(f(a)) = g(f(a')), because g is a function.
Thus, a = a' because g(f(a)) = a and g(f(a')) = a'.

We then prove that f is a surjection; that is, we prove that for each b ∈ B, there exists a ∈ A such that f(a) = b.

Let a = g(b). Then, f(a) = f(g(b)) = b.

We now prove that g = f^{-1}. Consider an arbitrary a ∈ A. If f(a) = b ∈ B, then according to (1), g(b) = g(f(a)) = a. Conversely, for an arbitrary b ∈ B, if g(b) = a, then f(a) = f(g(b)) = b. Thus, this implies that g(b) = a iff f(a) = b; thus, g = f^{-1}. By the symmetry between f and g, we can similarly prove that g is a bijection and f = g^{-1}. Thus, the theorem is proved.

When two functions f: A → B and g: B → A are inverses of each other, we have the relationship g ∘ f(a) = a for all a ∈ A, or, equivalently, g ∘ f = I$_A$, where I$_A$ denotes the identity function defined on A (i.e., I$_A$ (a) = a for all a ∈ A). Similarly, we have f ∘ g = I$_B$, the identity function defined on B.

Computing the inverse of a mathematical function

There is a simple way to compute inverses of mathematical functions. Whenever you have a function such as

$y = f(x) = 2x + 1,$

Place an x where ever you see a y, and a y where ever you see a x. Then, solve for y as follows:

$y = 2x + 1$

Inverse function is

$x = 2y + 1$
$2y = x - 1$
$y = (x - 1)/2.$

Computing the inverse of a function composition

If $f: A \to B$ and $g: B \to C$ are two bijections.
Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Since both f and g are bijections, $g \circ f$ is also a bijection and its inverse exists. To prove $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, it suffices to prove that

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C \quad \text{and} \quad (f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$$

$$
\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1}) &= ((g \circ f) \circ f^{-1}) \circ g^{-1}, \\
&= (g \circ (f \circ f^{-1})) \circ g^{-1}, \text{ associative} \\
&= (g \circ I_B) \circ g^{-1} \\
&= g \circ g^{-1}, \text{ defn. of identity} \\
&= I_C,
\end{aligned}
$$

Let f(x) = 2x+1 and g(x)=3x+4 and the domains and ranges for both function are the real numbers. So, we can compute
$$f^{-1}(x) = (x - 1)/2 \qquad g^{-1}(x) = (x - 4)/3$$

If we want to find $(g \circ f)^{-1}$, we can compute it as $f^{-1} \circ g^{-1}(x) =$

$$
\begin{aligned}
f^{-1}(g^{-1}(x)) = f^{-1}((x - 4)/3) &= ((x - 4)/3 - 1)/2 \\
&= (x - 4)/6 - 3/6 \\
&= (x - 7)/6
\end{aligned}
$$

If A and B are two *finite* sets, we can make the following claims:

If there is a injection $f: A \to B$, then $|A| \leq |B|$
If there is a surjection $g: A \to B$, then $|A| \geq |B|$
If there is a bijection $h: A \to B$, then $|A| = |B|$.

If $f: A \to B$ is an injection, and if we enumerate the elements of A as $A = \{a_1, a_2, \ldots, a_n\}$, where $n = |A|$, then the images $f(a_1), f(a_2), \ldots, f(a_n)$, are mutually distinct because f is one-to-one. Thus, $|B| \geq n$ because B contains these n elements.

Similarly, if $g: A \to B$ is a surjection, and if we enumerate the elements of B as $B = \{b_1, b_2, \ldots, b_m\}$, then A must contain the pre-images of the elements of B. Suppose we use the notations $a_1, a_2, \ldots, a_m$ to denote, respectively, the pre-images of $b_1, b_2, \ldots, b_m$; thus, $g(a_i) = b_i$ for $1 \leq i \leq m$. Since g is a function, the elements $a_1, a_2, \ldots, a_m$ must be mutually distinct (i.e., if $a_i = a_j$ for some $i \neq j$, then $b_i = g(a_i) = g(a_j) = b_j$, a contradiction). Therefore, $|A| \geq m = |B|$.