A Brief Introduction to Countability

Countability is covered thoroughly in COT 4210, so these notes simply serve as a quick introduction to the topic.

We define a set as infinitely countable if we can create a one-to-one correspondence between the natural numbers (1, 2, 3, …) and the set.

Basically, what this means is that, if you pick an arbitrary element in the set you are trying to show is countable, I should be able to tell you where it appears in an ordered list of items of that set.

Let's take a look at an easy example:

The set of non-negative integers is countable. We make the one-to-one correspondence as follows:

Rank	Non-negative integer
1	0
2	1
3	2
…	…

For example, if you ask me what the 10th number on my list is, I can tell you it's 9. Similarly, if you ask me where is the non-negative integer 19 on my list, I can tell you it appears as the 20th number. If we can do this both ways for arbitrary values, then the second list is countable. (In this example, the kth number on the list is k-1.)

Now, let's prove that the set of integers is countable. One might start out with the mapping above…but the problem is…

WHERE DOES -1 SHOW UP ON THE LIST?

Unfortunately, if we start our list 0, 1, 2, 3, …, we can't answer that question. Here is a better ordering that makes sure we hit every integer eventually:

Rank	**Integer**
1	**0**
2	**1**
3	**-1**
4	**2**
5	**-2**
6	**3**
7	**-3**

The kth value on this list is k/2 if k is even and -(k-1)/2, if k is odd. Similarly, if we want to know where the integer k is on the list, if k is positive, it appears at position 2k. If k is negative, it appears at position -2k+1. For our purposes, there is no need to formalize the relationship. Rather, showing the chart and arguing that any arbitrary item on the second list will appear at some specified point is good enough.

Are Fractions Countable?

One might think that fractions aren't countable. After all, there are WAY more fractions than positive integers!!! (Each integer is a fraction, but there are fractions that aren't positive integers.)

Hopefully you can recognize the flaw in this logic, just based on the previous example. In the previous example, it seems that there are "more" integers than positive integers, since every positive integer is an integer, but there are other integers that aren't positive integers, yet the cardinality of both sets is the same, since we were able to do the one to one mapping. Here is how we show that fractions are countable. Instead of making a table, we make a 2D grid, but show a special way to order the items in the grid:

N/D	1	2	3	4	5	6	7
0	0/1[1]	0/2[3]	0/3[6]	0/4[10]	0/5[15]	0/6[21]	0/7[28]
1	1/1[2]	1/2[5]	1/3[9]	1/4[14]	1/5[20]	1/6[27]	1/7
-1	-1/1[4]	-1/2[8]	-1/3[13]	-1/4[19]	-1/5[26]	-1/6	-1/7
2	2/1[7]	2/2[12]	2/3[18]	2/4[25]	2/5	2/6	2/7
-2	-2/1[11]	-2/2[17]	-2/3[24]	-2/4	-2/5	-2/6	-2/7
3	3/1[16]	3/2[23]	3/3	3/4	3/5	3/6	3/7
-3	-3/1[22]	-3/2	-3/3	-3/4	-3/5	-3/6	-3/7

The numerators are highlighted in yellow and the denominators in green. All possible fractions appear somewhere in the grid. The way we will count them is in a diagonal fashion. Assign the first fraction to be 0/1, the second to be 1/1, the third to be 0/2, etc. The numbering system is shown in red. Basically, you read successive diagonals going from the bottom left to top right. Notice, that in this fashion, if you ask me when I'll get to a particular fraction, say -23/77, I can tell you. Similarly, if you ask me which fraction is 27^{th} on my list, I can tell you. Thus, this proves that the set of fractions is countable, since we've established a one to one correspondence between positive integers and the fractions.

Is the Cartesian Product of Two Countable Sets Countable?

Let A and B be two countable sets. Then, there exists a one to one correspondence between the positive integers and each set. Let the items of set A, in order, be a_1, a_2, a_3, … Similarly, let the items of set B, in order, be b_1, b_2, b_3, … We can use the same type of argument as with the fractions to show that A x B is countable. Make a grid:

A/B	b_1	b_2	b_3	b_4	b_5	b_6	b_7
a_1	$(a_1,b_1)^{1}$	$(a_1,b_2)^{3}$	$(a_1,b_3)^{6}$	$(a_1,b_4)^{10}$	$(a_1,b_5)^{15}$	$(a_1,b_6)^{21}$	$(a_1,b_7)^{28}$
a_2	$(a_2,b_1)^{2}$	$(a_2,b_2)^{5}$	$(a_2,b_3)^{9}$	$(a_2,b_4)^{14}$	$(a_2,b_5)^{20}$	$(a_2,b_6)^{27}$	(a_2,b_7)
a_3	$(a_3,b_1)^{4}$	$(a_3,b_2)^{8}$	$(a_3,b_3)^{13}$	$(a_3,b_4)^{19}$	$(a_3,b_5)^{26}$	(a_3,b_6)	(a_3,b_7)
a_4	$(a_4,b_1)^{7}$	$(a_4,b_2)^{12}$	$(a_4,b_3)^{18}$	$(a_4,b_4)^{25}$	(a_4,b_5)	(a_4,b_6)	(a_4,b_7)
a_5	$(a_5,b_1)^{11}$	$(a_5,b_2)^{17}$	$(a_5,b_3)^{24}$	(a_5,b_4)	(a_5,b_5)	(a_5,b_6)	(a_5,b_7)
a_6	$(a_6,b_1)^{16}$	$(a_6,b_2)^{23}$	(a_6,b_3)	(a_6,b_4)	(a_6,b_5)	(a_6,b_6)	(a_6,b_7)
a_7	$(a_7,b_1)^{22}$	(a_7,b_2)	(a_7,b_3)	(a_7,b_4)	(a_7,b_5)	(a_7,b_6)	(a_7,b_7)

Then, we order these ordered pairs by going up each diagonal, from bottom left to top right, successively. Any arbitrary term in the Cartesian product will show up somewhere on this grid. Similarly, if given a number k, we can produce the kth element of the Cartesian product. Thus, the Cartesian product of two countable sets is ALSO countable.

Note: In both of these last two examples, had we numbered the top row 1, 2, 3, …, we would never get to the second row, and that mapping would NOT prove that either set is countable. Note that this proves that just because you come up with an invalid ordering of a set doesn't mean that it is not countable. We need an entirely different technique to prove a set isn't countable…which you will learn in COT 4210. These are just a couple reasons why countability is quite counter-intuitive!!!