

Some Sample Counting Questions

1. A spider wears 8 socks and 8 shoes on his 8 feet. For any individual foot, he must put the sock on before the shoe. In how many different orders can the spider put on his socks and shoes? (Note: for example, for a human, there are 6 possible ways:

1. sock left, sock right, shoe left, shoe right
2. sock left, sock right, shoe right, shoe left
3. sock right, sock left, shoe left, shoe right
4. sock right, sock left, shoe right, shoe left
5. sock left, shoe left, sock right, shoe right
6. sock right, shoe right, sock left, shoe left)

Solution

We can think of simply noting which foot each event occurs. For example, consider the following list, which omits what is being placed on the foot, but just the foot on which an item is placed:

1, 3, 2, 1, 3, 8, 6, 6, 4, 5, 2, 7, 7, 8, 5, 4

Do we know at which step a shoe was placed on foot 4?

YES!!! 4 is listed twice, 9th and 16th on the list. But, we know that the sock has to come before the shoe, so the spider must have put his sock on his fourth foot on the 9th step and the shoe on his fourth foot on the 16th step!

This means that each permutation of the symbols above represents the number of ways the spider can put on its socks and shoes. Using the permutation formula, we have $\frac{16!}{(2!)^8} = \frac{16!}{2^8}$ ways the spider can put on its socks and shoes.

2. There is a bowling tournament where the 5th ranked bowler plays the first match against the 4th ranked bowler. The loser gets 5th place in the tournament and the winner goes on to play the 3rd ranked bowler. The loser of that match gets 4th place in the tournament and the winner of that match gets to play the 2nd ranked bowler in the next match. The loser of this match finishes 3rd in the tournament and the winner goes onto play the 1st ranked bowler. In this last match, the loser gets 2nd place and the winner gets 1st place. In how many different orders can the bowlers finish?

Solution

There are 4 matches, each match can either be a win or a loss for the challenger. So, there are 2^4 possible outcomes of the matches. Each of these must lead to a different ordering of the contestants. Thus, the answer is just 16 possible orders. For any particular listing of wins and losses, you can compute the unique ordering. For example, if the matches go W, W, L, L, then the finishing order is 4th seed, 3rd seed, 5th seed, 2nd seed and 1st seed.

3. How many positive odd integer solutions are there to the equation,
 $a + b + c + d = 98$?

Solution

Since a , b , c and d must be odd, let $a = 2a' + 1$, $b = 2b' + 1$, $c = 2c' + 1$ and $d = 2d' + 1$, where a' , b' , c' , d' are all non-negative integers.

$$(2a'+1) + (2b'+1) + (2c'+1) + (2d'+1) = 98$$

$$2a' + 2b' + 2c' + 2d' = 94$$

$$a' + b' + c' + d' = 47$$

Now, we've reduced our problem to a regular combinations with repetition problem with $n = 47$ and $r = 4$, which has $\binom{50}{3}$ solutions. Each solution to the equation above maps to a unique solution to the given equation and vice versa.

4) How many 15 letter arrangements of 5 A's, 5 B's and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters and no C's in the last 5 letters?

Solution

We want no A's in the first five letters, no B's in the next five letters and no C's in the last five letters.

Let's first try to fill the first five letters. We must only have B's and C's. We can choose the following:

- 0 B's, 5 C's
- 1 B, 4 C's
- ...
- 5 B's 0 Cs

Let's say we choose k B's for the first five slots. We can do this in $\binom{5}{k}$ ways. Then, the C's are forced into the other slots.

Now, the remaining $5-k$ B's must go in slots 11 through 15. Thus, we can choose the location for these B's in $\binom{5}{5-k} = \binom{5}{k}$ ways.

Finally, we must fill the middle 5 slots with A's and C's. Specifically, we have already placed $5 - k$ C's in the first 5 slots and placed k A's in the last five slots. So, really, we can just choose k out of 5 slots for the As in $\binom{5}{k}$ ways, and then the C's are forced.

We can break up our counting into 6 batches for each possible value of k , the number of B's in the first five slots. Thus, the answer is

$$\sum_{i=0}^5 \binom{5}{k}^3 = 1^3 + 5^3 + 10^3 + 10^3 + 5^3 + 1^3 = 2252.$$

5) Call a number "prime-looking" if it is composite but not divisible by 2, 3 or 5. The three smallest prime-looking numbers are 49, 77 and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are less than 1000?

Solution

Numbers can be in these categories:

- 1) Prime (size 168)
- 2) Divisible by 2, 3 or 5, BUT NOT PRIME
- 3) Composite Looking
- 4) 1 (size 1)

All 1000 positive integers from 1 to 1000 must fit in one of these 4 categories. What we know about each set's size is written above in yellow.

Strategy: Find out how many items in set #2 and then use subtraction to figure out the final answer.

Count all values divisible by 2, 3 or 5 via the inclusion-exclusion principle

$$\left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor$$

=

$$500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$$

The number of values in group 2 is $734 - 3 = 731$, because we have to subtract out 2, 3 and 5, which are not part of group 2.

$$\text{Final answer} = 1000 - 168 - 731 - 1 = \underline{100}$$

6) A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 00050. If the odometer now reads 002005, how many miles has the car actually traveled?

Solution

Basically, our odometer has 9 possible digits for each slot, not 10. So, it's like a number in base 9... 2005 really means that the 4th slot from the right moved twice, and it moves twice with the other slots to the right of it go through 9^3 positions. So after the car has traveled $2 \times 9^3 = 1458$ miles, the odometer reads 002000. Then the odometer counts 002001, 002002, 002003 and 002005. It goes four more miles to get to 002005, so the total miles traveled is $1458 + 4 = \underline{1462}$.

7) Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \dots, 12\}$, including the empty set, are spacy?

Solution

Combinations with repetition tells us that the equation

$x_1 + x_2 + \dots + x_r = n$ has exactly $\binom{n + r - 1}{r - 1}$ non-negative integer solutions for (x_1, x_2, \dots, x_r) .

First observation: we can't have more than 4 items from the set, since after you take an item you have to skip 2 more after it.

Sets of size 0, 1, 2, 3, and 4

Size 0: 1 set (empty set)

Size 1: 12 sets (any of the 12 sets of 1 item are spacy)

Size 2: We pick our 2 items but then have to separate them by at least 2:

(not taken bin #1)X (not taken bin with 2 already, bin #2) X (not taken bin bin #3)

The x's are 2 items and we also have to have 2 items in between the x's. This means that we have 8 more items free to place in the 3 non-taken bins.

Let x be the size of bin 1, y be the size of bin 2 and z be the size of bin 3. We know that

$$x + y + z = 8, \text{ and } x, y \text{ and } z \text{ are non-negative integers.}$$

There are $\binom{8+3-1}{3-1} = \binom{10}{2} = 45$ ways to place these 8 items. So there are precisely 45 spacey sets with 2 items in them.

Consider the following solution to the equation:

$$x = 4, y = 1 \text{ and } z = 3$$

This means that {1,2,3,4} are NOT in the set, since $x = 4$
5 is in the set

This means that {6,7,8} are NOT in the set.

9 is in the set

This means that {10,11,12} are NOT in the set, since $z = 3$

Size 3: We pick our 3 items but then have to separate them by at least 2:

$$(\text{bin \#1}) \quad X \quad (\text{bin \#2}) \quad X \quad (\text{bin \#3}) \quad X \quad (\text{bin \#4})$$

Two things must be placed in both bin #2 and bin #3. So we have already accounted for 3 X's and 4 spacing items, leaving 5 other times to be placed amongst 4 bins, which can be done in $\binom{5+4-1}{4-1} = \binom{8}{3} = 56$ spacy subsets of size 3.

Size 4: We pick our 4 items but then have to separate them by at least 2:

$$(\text{bin \#1}) \quad X \quad (\text{bin \#2}) \quad X \quad (\text{bin \#3}) \quad X \quad (\text{bin \#4}) \quad X \quad (\text{bin \#5})$$

Two things must be placed in both bin #2, bin #3 and bin #4 So we have already accounted for 4 X's and 6 spacing items, leaving 2 other times to be placed amongst 5 bins, which can be done in $\binom{2+5-1}{5-1} = \binom{6}{4} = 15$ spacy subsets of size 4.

Final answer = $1 + 12 + 45 + 56 + 15 = 13 + 101 + 15 = \underline{129}$.

