

Example Counting Questions

(All taken from past Foundation Exams)

Martian Life

Intelligent life has finally been discovered on Mars! Upon further study, linguistic researchers have determined several rules to which the Martian language adheres:

- 1) The Martian alphabet is composed of three symbols: a, b, and c.**
- 2) Each word in the language is a concatenation of four of these symbols.**
- 3) Each command in the language is composed of three words.**

a) How many distinct commands can be created if words in a single command can be repeated and two commands are identical only if the three words AND the order in which the words appear are identical? (Thus, the commands "aaca baaa aaca" and "baaa aaca aaca" are two DIFFERENT commands.)

Total of 12 symbols in a command. For each of these symbols, we have 3 choices without any restrictions. These choices are independent of one another, so the total number of commands we have is 3^{12} .

b) How many distinct commands can be created if word position does not affect meaning and a given word may appear at most once in a single command? (Thus, "abca bbac abbb" and "bbac abbb abca" should NOT count as different commands, and "aaca baaa aaca" is an INVALID command.)

Since we are not allowed to repeat words and word order doesn't matter, we are essentially choosing three words out of the a possible number of words. Thus, we must first figure out the possible number of words. There are three choices for each of four symbols, using the multiplication principle as we did in part a, we have $3^4 = 81$ possible Martian words. Of these, we can choose three to make a valid command. Thus, the total number of possible commands here is ${}_{81}C_3 = (81)(80)(79)/6 = 85320$

Counting Numbers

Consider six-digit numbers with all distinct digits that do NOT start with 0. Answer the following questions about these numbers. Leave the answer in factorial form.

- a) How many such numbers are there?**
- b) How many of these numbers contain a 3 but not 6?**
- c) How many of these numbers contain either 3 or 6 (or both)?**

a) There are 9 choices for the first digit, and then 9 choices for the second digit (0 has been added as a choice), 8 for the third, 7 for the fourth, 6 for the fifth, and 5 for the sixth. Total = $(9)(9)(8)(7)(6)(5) = 9(9!)/4! = 136080$.

b) We need to separate the counting into two categories

1) 3 is the first digit

2) 3 is NOT the first digit

For the first category, we have one choice for the first digit, followed by 8 choices for the second digit (not 3 or 6), 7 choices for the third digit, 6 choices for the fourth digit, 5 choices for the fifth digit and 4 choices for the sixth digit.

$$\text{Total} = (8)(7)(6)(5)(4) = 8!/3!$$

For the second category, we have 7 choices for the first digit (not 0, 3, or 6), now we must guarantee that a 3 is picked. There are five PLACES to place the 3. For the remaining 4 digits, we have 7 choices, 6 choices, 5 choices and 4 choices, respectively for each of these. (To see this, imagine the 3 was placed 2nd. Then for the third digit you could choose any number except for the first digit, 3 and 6. Similarly, no matter where the 3 is placed, you always have 7 choices for the next placed digit, then 6, etc.)

$$\text{Total} = (7)(5)(7)(6)(5)(4) = 35(7!)/3!$$

The total of both of these categories is $8!/3! + 35(7!)/3! = 36120$

c) Count the number of numbers that contain neither:

There are 7 choices for the first digit (not 0, 3 or 6), 7 choices for the second digit, 6 choices for the third digit, 5 choices for the fourth digit, 4 choices for the fifth digit and 3 choices for the sixth digit. Total = $(7)(7)(6)(5)(4)(3) = 7(7!)/2!$

Now, the answer to the question given is the value above subtracted from the answer in part a. Thus, this answer is $9(9!)/4! - 7(7!)/2! = 118440$.

Permutations

a) How many distinguishable ways are there to rearrange the letters in the word COMBINATORICS?

b) How many distinguishable arrangements are possible with the restriction that all vowels (“A”, “I”, “O”) are always grouped together to form a contiguous block?

c) How many distinguishable arrangements are possible with the restriction that all vowels are alphabetically ordered and all consonants are alphabetically ordered? For example: BACICINOONRST is one such arrangement.

a) There are 13 letters in the word COMBINATORICS, including three duplicates, two C's, two O's and two I's. So, the total number of arrangements is $13!/(2!)^3$.

b) If all five vowels are consecutive, they form a single block. Then first we need to count permutations of the consonants and one block of vowels. Given eight consonants with one duplicate (two C's), we have $9!/2!$. But every arrangement of consonants and the block of vowels can be combined with any permutation of vowels inside the block. For five vowels including two duplicates we have $5!/(2!)^2$ possible permutations inside the block. Then by the product rule we get the answer: $(9! \cdot 5!)/(2!)^3$.

c) Any arrangement is completely defined by specifying which 5 of 13 positions should be occupied by vowels (or equivalently which 8 out of 13 should be occupied by consonants). So we just need to count the number of ways to select 8 positions out of 13 (or equivalently 5 positions out of 13), that is $13!/(8!5!)$. Given any such selection, both consonants and vowels are distributed alphabetically into assigned slots.

More Permutations

How many 6-letter words can be formed by ordering the letters *ABCDEF* if *A* appears before *C* and *E* appears before *C*?

Under given restrictions there are two possible arrangements for letters A, C and E between themselves: either A appears before E, or E before A, i.e. AEC or EAC, so we have two choices for this task. After that we can choose 3 slots to place letters A, C and E out of 6 possible in a 6-letter word. If the order of A, C and E is fixed, we count selections $C(6, 3)$. After we fill 3 slots with letters A, C and E, we can make $3!$ permutations of the letters B, D and F using remaining 3 slots. By the product rule the total number of ordering will be $2 \cdot C(6, 3) \cdot 3! = 2 \cdot 6 \cdot 5 \cdot 4 = 240$.

Even More Permutations

a) How many permutations of the word **FOUNDATION** are there?

$10!/(2!2!)$, since there are 10 letters total with 2Os and 2Ns.

b) A valid password is 5 letters long and uses a selection of the letters in the word **FOUNDATION**. (Thus, a password may have at most 2 N's, at most 2 O's, and at most 1 copy of each of the other letters {A, D, F, I, U, T} in **FOUNDATION**.) How many valid passwords are there?

Split up the counting into three separate categories:

- 1) Passwords with 5 distinct letters*
- 2) Passwords with 4 distinct letters*
- 3) Passwords with 3 distinct letters*

- 1) *We have 8 distinct letters to choose from and we are choosing 5. There are $P(8,5) = 8!/3!$ ways to do this.*
- 2) *We first choose either two Ns or two Os. This can be done in 2 ways. Then we choose three distinct letters out of the 7 remaining. We can choose the three letters in $C(7,3)$ ways. Thus, we have $C(7,3) \times 2 = 70$ ways to choose our letters. Each of these choices gives rise to $5!/2! = 60$ permutations.*
- 3) *We have to choose all Ns and Os, which leaves us one choice out of the remaining 6 letters. We can choose this letter in 6 ways. For each of these choices, we can make $5!/(2!2!) = 30$ permutations. So there is a total of 180 permutations of this kind to count.*

Adding up, we get the total number of valid passwords to be

$$8!/3! + 70 \times 60 + 180 = 6720 + 4200 + 180 = 11100.$$

c) An ascending word is one where the letters of the word appear in alphabetical order. Thus, for example, AFNNT is a valid password that is ascending. How many of the valid passwords are ascending?

Using the logic from above, but just counting combinations, we get the answer to be $C(8,3) + 70 + 6 = 56 + 70 + 6 = 132$.

Note: The reason we only count combinations here is that each distinct combination of letters gives rise to exactly one valid password. (For each set of letters, there is one and only one way to alphabetize them.)

Playing Cards

You are given 12 playing cards that include 3 spades, 3 hearts, 3 diamonds, and 3 clubs (that is, 3 cards of each “suit.”)

(a) How many ways can one select 6 cards total from the 12 when choosing 1 spade, 1 heart, 2 diamonds, and 2 clubs?

(b) How many ways can one select 6 cards total from the 12 when choosing at least 1 card from each suit?

(a) $\binom{3}{1}\binom{3}{1}\binom{3}{2}\binom{3}{2} = 3^4 = 81.$

(b) *The choices may be distributed among the suits either as 1 from one suit, 1 from another suit and 2 from each of the remaining two suits (1,1,2,2); or they may be distributed as 1,1,1,3 for a total of 6 cards where at least one comes from each suit. There are $\binom{4}{2} = 6$ ways to decide which two suits contribute 1 card in the 1,1,2,2 cases and there are $\binom{4}{1} = 4$ ways to choose which suit contributes 3 cards in the 1,1,1,3 cases. Thus, the answer is $6\binom{3}{1}\binom{3}{1}\binom{3}{2}\binom{3}{2} + 4\binom{3}{1}\binom{3}{1}\binom{3}{1}\binom{3}{3} = 594.$*

3) An ice cream shop lets its customers create their orders. Each customer can choose up to four scoops of ice cream from 10 different flavors. In addition, they can add any combination of the 7 toppings to their ice cream. (Note: Please leave your answer in factorials, combinations, and powers.)

a) If a customer is limited to at most two scoops of the same flavor, how many possible orders with exactly 4 scoops and up to 5 toppings can the customer make? (Assume each order has at least one topping.)

b) Suzanne wants to make 7 separate orders for ice cream. Each order will have exactly 1 scoop and 1 topping. If no flavor or topping is requested more than once, how many combinations of orders can Suzanne make?

a) If we ignore toppings initially, we have a problem of combinations WITH repetition. We are choosing 4 items from 10 possible items, allowing for repetition. This can be done in $C(4+10-1,4) = 715$. BUT, here we are counting choices that have 3 and 4 scoops of the same flavor. We need to subtract these out. So, our next sub-problem becomes to count the number of ways we can order exactly 4 scoops with one flavor repeated at least 3 times. Since only ONE flavor can be repeated at least 3 times, pick this flavor. There are 10 choices for it. Go ahead and pick 3 scoops of this flavor. Now you are left with 1 scoop to pick out of the 10 total flavors. This can be done in 10 ways as well. Thus, there are a total of $10 \times 10 = 100$ combinations of scoops with one flavor repeated at least 3 times. So we have $715 - 100 = 615$ ways to choose the scoops of ice cream.

Now, the choice of toppings is independent from the scoops. There are total of 2^7 total combinations of toppings we can receive without restrictions. BUT, we are only allowed to get up to 5 toppings, but at least one topping. Thus we just subtract out the number of ways to get 0, 6 or 7 toppings. There is $C(7,0) = 1$ way to get zero toppings, $C(7,6) = 7$ ways to choose 6 toppings, and $C(7,7)=1$ way to choose all 7 toppings. So there are a total of $2^7 - 1 - 7 - 1 = 119$ ways to choose the toppings.

This gives us a final answer of $615 \times 119 = 73185$ possible orders for the customer.

b) This question is the same as how many injections are there from a set of size 7 to a set of size 10. Imagine the domain being the toppings. Since we are forced to pick each topping exactly once, and none of the flavors are repeated, we are mapping each topping to a distinct element from the co-domain, the set of flavors. We can do this in $P(10,7)$ ways. $P(10,7) = 10!/3! = 604800$.

Ping-Pong Balls

Tom has 15 ping-pong balls each uniquely numbered with a number between 1 and 15. He also has a red box, a blue box, and a green box. (a) How many ways can Tom place the 15 distinct balls into the three boxes? (b) Suppose now that Tom has placed 5 ping-pong balls in each box. How many ways can he choose 5 balls from the three boxes so that he chooses at least one from each box?

(a) Each solution is a function that maps 15 elements into 3. Thus, the answer is 3^{15} .

(b) The 5 balls can be chosen either as 1,1,3 (1 from a box, 1 from another box, 3 from remaining box) or as 1,2,2. There are 3 ways to select as 1,1,3 (take the 3 balls from red or 3 from blue or 3 from green). There are 3 ways to select as 1,2,2. Thus, recalling that the balls are uniquely numbered, the answer is

$$3 * C(5,1) * C(5,1) * C(5,3) + 3 * C(5,1) * C(5,2) * C(5,2) = 2250.$$