Combinations with Repetition

Consider the following problem:

You are buying some beer for a gathering. In particular, you must buy from the selection of six packs listed below:

Miller Lite, Bass, Natty Light, Sam Adams, and Killians

You must choose exactly 8 six packs to buy. How many different combinations of beer can you buy?

Imagine solving the problem as follows:

You have five (labeled) bags that you carry your beer in - one for each possible beer. Each possible combination can be listed as a 5-tuple. For example, the 5-tuple (3, 2, 1, 1, 1) stands for 3 six packs of Miller Lite, 2 six packs of Bass, 1 six pack of Natty, 1 six pack of Sam, and 1 six pack of Killians. In essence, we want to count how many different 5-tuples we can form, such that sum of the 5 numbers is 8. So, we can reduce our problem into solving the following:

How many non-negative integer solutions are there to the equation

$x_1 + x_2 + x_3 + x_4 + x_5 = 8$?

The answer to each of these questions is actually a combination in disguise. Another way to look at this problem is to count the number of ways you can divide a set of 8 stars by 4 bars, or barriers. Consider this one way:

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This corresponds to the combination listed above. Each bar tells us when we are starting the next bag so to speak. We KNOW that we must have exactly 8 items, (hence the 8 stars), and we know that we must fit them in exactly 5 bags, which we denote with 4 bars. (Why do we use 4?)

As you can see, every possible combination of orders has a corresponding set of bars and stars. Similarly, each unique ordering of bars and stars corresponds to one specific meal order.

Thus, we can answer our question by finding the number of ways we can arrange 8 stars and 4 bars. But this is the type of question we answered before!!! We can look at it from 2 perspectives:

1) Let all stars be designated by an A, and all bars with a B. The question then becomes how many permutations are there of the words AAAAAAAABBBB. From last time, the answer is $12!/(8!4!) = {}_{12}C_8$.

2) There are 12 total locations to place our stars and bars. But, really, all we have to do is place the stars and the location of the bars is fixed. Thus, we are choosing 8 of the 12 possible locations for our stars. This can be done in ${}_{12}C_8$ ways.

In general, the answer to the following questions:

1) How many ways can you choose n items, where you have r different choices for each item?

2) How many ways can you place n identical objects in r labeled bins?

3) How many non-negative integer solutions are there to the equation $x_1 + x_2 + ... + x_r = n$?

are all the same. In particular the answer is the number of ways you can permute n stars and r – 1 bars. This can be done in $_{n+(r-1)}C_n$ or $_{n+(r-1)}C_{r-1}$ ways. (Note that both of these expressions are the exact same. Can you figure out why?)

<u>Sample Problem</u>
In how many ways can you give 10 dollar bills to 4 children?

In this problem, the number objects we are distributing is n = 10, and the bins are the children, so the number of bins r = 4. Using the formula we just derived, we can distribute the bills in $\binom{10+4-1}{4-1} = \binom{13}{3}$ ways.

<u>Dealing with a Minimum Restriction</u>
Consider the following closely related problem:

In how many ways can you give 10 dollar bills to 4 children, given that we must give at least one dollar to each child?

Then we can do the following. First give one dollar to each of the four kids. Then you have 6 dollars to freely distribute amongst the 4 kids. This can be done in $_{6+(4-1)}C_6 = 84$.

A Single Maximum Restriction

What if you may not give any more than 4 dollars to the oldest child? (But do not have to give all of the kids at least one dollar...)

The total number of ways to distribute the money from before is ${}_{13}C_3$. From this, we must subtract the number of ways that the oldest child receives 5 or more dollars.

We can count these ways in the following manner. Give the oldest child 5 bucks. Then, we are free to distribute the rest of the 5 dollars to the 4 kids in any way we want. This can be done in ${}_{5+(4-1)}C_5$ ways. Thus, our final answer is ${}_{13}C_3$ - ${}_{8}C_3$ = 230.

Multiple Maximum Restrictions

Let's say we want to buy 30 items from Taco Bell, and are choosing from the following items:

Hard Shell Taco, Soft Shell Taco, Bean Burrito, 7 Layer Burrito, Chicken Burrito, Mexican Pizza and Nachos. Unfortunately, there are only 4 more Mexican Pizzas in stock and 7 Chicken Burritos in stock.

In how many ways can you purchase the items?

Without any restrictions, we have n = 30 objects to distribute amongst r = 7 bins, which we can do in $\binom{30+7-1}{7-1} = \binom{36}{6}$ ways.

But, we must subtract out of this all combinations with 5 or more Mexican Pizzas. To count these, let's buy 5 Mexican Pizzas, leaving us with 25 more items to buy out of 7 types of objects, which we can do in $\binom{25+7-1}{7-1} = \binom{31}{6}$ ways.

Now, we must also subtract out all the combinations with 8 or more Chicken Burritos. To calculate this, buy 8 chicken burritos right away, leaving 22 items left to buy out of 7 types. We can do this in $\binom{22+7-1}{7-1} = \binom{28}{6}$ ways.

So, we might think the answer is $\binom{36}{6} - \binom{31}{6} - \binom{28}{6}$ ways. Unfortunately, when we subtracted out combinations with 5 or more Mexican Pizzas, some of those ALSO had 8 or more Chicken burritos. In fact, we subtracted all combinations with 5 or more Mexican Pizzas AND 8 or more Chicken Burritos out twice. Thus, by the Inclusion-Exclusion Principle, we must add these back in.

The total number of combinations with at least 5 Mexican Pizzas and 8 Chicken burritos can be calculated by buying these 13 items right away, leaving 17 items to buy out of 7 types of items, which can be done in $\binom{17+7-1}{7-1} = \binom{23}{6}$ ways.

Thus, the correct answer is $\binom{36}{6} - \binom{31}{6} - \binom{28}{6} + \binom{23}{6}$.

<u>Inequalities</u>
What if, instead of buying precisely 30 items out of 7 types, we wanted to know the number of ways to buy 30 OR fewer items out of 7 types?

This seems like a very difficult problem!

One approach would be to buy 0 items, 1 item, then 2 items, then 3 items, and then add up all 31 of these answers.

This seems a bit tedious, let's see if there is another way.

We would typically map the problem to the number of non-negative integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 30$$

Now, imagine that we choose not to buy 30 items. What we *could* do is assign all the items we *didn't buy* to a "garbage bin". Call the number of items in this garbage bin, x_8. Then, what we really seek is the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 30$$

In fact, there is a one-to-one correspondence between all possible non-negative integer solutions to this equation right above, and the former inequality.

Try it!

Consider this solution (3, 8, 2, 6, 1, 1, 4, 5)… This is in one to one correspondence with $x_1 = 3$, $x_2 = 8$, $x_3 = 2$, $x_4 = 6$, $x_5 = 1$, $x_6 = 1$, and $x_7 = 4$.

Similarly, given any solution to the inequality, we can calculate the unique correct solution for the equality below it. Thus, the number of solutions to each is the same.

Thus, in general, the number of non-negative integer solutions to: $x_1 + x_2 + x_3 + \ldots + x_r \leq n$ is just $\binom{n+r}{r}$.