

Combinations

Combining our result for counting combinations, some logic, the sum rule and the product rule, we can handle more sophisticated counting questions.

Furthermore, what we find out is that depending on how you view a counting question conceptually, the amount of work to get to the same answer (that looks different) varies greatly.

In addition, there are many counting questions that are really just combinations in disguise.

Finally, with combinations, we can do both algebraic proofs and combinatorial proofs. The former you are likely familiar with, but the latter are fairly different; they involve making arguments in English to show a one to one correspondence between two different sets. This latter skill is critical in becoming better at solving counting problems.

In this lecture, we'll investigate all of these qualities of combinations. It's not a coincidence that the study of counting is called "combinatorics." Combinations ($\binom{n}{k}$) truly represent the backbone of the art of counting.

Combinations Problem Example

Let $S = \{1, 2, 3, \dots, 30\}$.

How many subsets A of S contain 5 elements, with 5 being the least?

In essence, we know that 5 must be one of our elements, so we are really free to choose only 4 elements. But, we have a restriction here too. We must choose those four elements from the set $\{6, 7, \dots, 30\}$. The number of ways to do this are ${}_{25}C_4$.

So this was a straight application of a combination after making a single observation.

Example Problem Building on the Previous One

How many subsets A of S contain 5 elements with the smallest element not being equal to 5?

We know the total number of subsets of S that are of size 5 is ${}_{30}C_5$. And we also know that of these, exactly ${}_{25}C_4$ have 5 as the smallest element. Thus, our answer should be the difference of these two, or ${}_{30}C_5 - {}_{25}C_4$.

Here, we added the subtraction principle to our previously derived combination.

A More Difficult Question?

How many subsets of S contain 5 elements with the smallest element less than 5?

This is actually quite a difficult question. The problem is that we don't know how many of the elements are less than 5. In fact we have 4 (disjoint) possibilities:

1 element is less than 5: Thus we choose 1 element from the set $\{1,2,3,4\}$ and 4 elements from the set $\{5,6,\dots,30\}$. Since these choices are independent, we can invoke the product rule to find the total number of ways to do this as ${}^4C_1 \cdot {}_{26}C_4$.

2 elements are less than 5: Thus we choose 2 elements from the set $\{1,2,3,4\}$ and 3 elements from the set $\{5,6,\dots,30\}$. Since these choices are independent, we can invoke the product rule to find the total number of ways to do this as ${}^4C_2 \cdot {}_{26}C_3$.

3 elements are less than 5: Thus we choose 3 elements from the set $\{1,2,3,4\}$ and 2 elements from the set $\{5,6,\dots,30\}$. Since these choices are independent, we can invoke the product rule to find the total number of ways to do this as ${}^4C_3 \cdot {}_{26}C_2$.

4 elements are less than 5: Thus we choose 4 elements from the set $\{1,2,3,4\}$ and 1 element from the set $\{5,6,\dots,30\}$. Since these choices are independent, we can invoke the product rule to find the total number of ways to do this as ${}^4C_4 \cdot {}_{26}C_1$.

Now, using the sum rule, we can add these values up to get the following answer:

$${}^4C_1 \cdot {}_{26}C_4 + {}^4C_2 \cdot {}_{26}C_3 + {}^4C_3 \cdot {}_{26}C_2 + {}^4C_4 \cdot {}_{26}C_1 = 76726$$

So, here we used multiple combinations, the product rule and the addition rule. Could we have solved it more easily?

A slightly easier approach

We have four cases:

Case 1: 1 is the smallest number. Then we must pick 4 numbers from the remaining 29. This can be done in ${}_{29}C_4$ ways.

Case 2: 2 is the smallest number. Then we must pick 4 numbers from the remaining 28. This can be done in ${}_{28}C_4$ ways.

Case 3: 3 is the smallest number. Then we must pick 4 numbers from the remaining 27. This can be done in ${}_{27}C_4$ ways.

Case 4: 4 is the smallest number. Then we must pick 4 numbers from the remaining 26. This can be done in ${}_{26}C_4$ ways.

So our answer is also

$${}_{29}C_4 + {}_{28}C_4 + {}_{27}C_4 + {}_{26}C_4 = 76726.$$

Probably the easiest approach

We have already shown that there are a total of ${}_{30}C_5$ subsets. Of these, we want to subtract out those whose smallest element is 5 or greater. We can count this group because it is just the number of subsets of the set $\{5, 6, 7, \dots, 30\}$, which is just ${}_{26}C_5$. Thus, another way to express the answer to this question is

$${}_{30}C_5 - {}_{26}C_5$$

I created this example to show the differences in difficulty of the same question, depending on how one breaks down/characterizes the counting to be done.

Odd/Even Sized Subsets of a Set

For any set, the number of subsets of even cardinality is the same as the number of subsets of odd cardinality. One way to look at this is the following. We know that this fact is true for a set of size 1. (There are two subsets, one with cardinality 0 and the other with cardinality 1.) Imagine adding one element to this set and then listing all the new subsets. For each “old” subset of even cardinality, we are adding a “new” subset of odd cardinality and vice versa. Thus, we maintain the same number of subsets of even and odd cardinality. Since a set of size n has 2^n subsets, the set has 2^{n-1} even subsets, and 2^{n-1} odd subsets.

Formally, this can be proven via induction $|S|$, the size of the set S for all non-empty sets. This is a good practice problem for everyone to do!!!

Combinatorial Proofs

A combinatorial proof is one where instead of doing lots of traditional math (say algebra), we make a conceptual counting argument to prove a claim.

Let's look at a simple example: Prove that for all positive integers k , $\frac{(2k)!}{2^k}$ is an integer.

Consider counting the number of permutations of the n symbols $x_1, x_1, x_2, x_2, \dots, x_k, x_k$. Using the formula derived above, we calculate this value to be: $(2k)!/(2!)^k$.

But we know that $2! = 2$, this the final value is

$(2k)!/2^k$, which MUST BE an integer since the number of permutations of a group of letters is ALWAYS an integer.

Pascal Triangle Identity - proved two ways

In high school, many students encounter what is known as Pascal's Triangle (typically taught in Algebra II in the United States), the first few rows of which looks like this:

				1						
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

It's likely that you learned the following things about this triangle:

- 1) Put 1's on the outsides of each row (first and last item)
- 2) To calculate any given item, just add the two numbers directly above it.
- 3) These are the numbers you use when expanding out $(x+y)^n$. For example, $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

However, for whatever reason, many students aren't taught that the entry in row n , item k on this table are simply equal to $\binom{n}{k}$, the number of ways to choose k items out of n .

Once we know that the latter is true, that means that what our high school teacher really taught us with rule #2 is that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

provided that $0 < k < n$. If $k = 0$ or $k = n$, then $\binom{n}{0} = \binom{n}{n} = 1$.

So, let's prove this identity!!!

Combinatorial Proof

First let's do the combinatorial proof.

Say we have n candies and we want to select k of them. Let one of the candies be a Snickers bar. There are $n-1$ candies that are not Snickers bars.

By definition, we can select our k candies out of n in $\binom{n}{k}$ ways.

Now, let's count these $\binom{n}{k}$ ways in a different manner. Let's split up our counting into two disjoint groups:

1) Sets of k candies that HAVE Snickers.

2) Sets of k candies that do NOT have Snickers.

Let's count the number of sets in category 1. Since we have to select a Snickers, we really only have $k-1$ candies left to select. We are selecting those $k-1$ candies from the $n-1$ choices remaining, so we can do this in $\binom{n-1}{k-1}$ ways, by definition of combination.

Now, let's count the number of sets in category 2. We have NOT selected Snickers, so we have to select ALL k candies from the remaining $n-1$ candies, which we can do in $\binom{n-1}{k}$ ways.

To visualize this, let's say $n = 5$, $k = 3$, with our candies being {Snickers, OhHenry, Hershey, M&Ms, Reece's}

Sets with Snickers

{S,O,H}

{S,O,M}

{S,O,R}

{S,H,M}

{S,H,R}

{S,M,R}

Sets in Yellow are all combinations of 2 items out of 4.

Sets without Snickers

{O,H,M}

{O,H,R}

{O,M,R}

{H,M,R}

Sets in Blue are all combos of 3 items out of 4 remaining.

Since the two categories are disjoint, AND they cover ALL sets of k candies out of n, it follows that the sum of these two equals $\binom{n}{k}$, giving us the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Now, let's look at an algebraic proof, where we start with the RHS and show it equals the LHS:

$$\begin{aligned}
 \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
 &= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{1}{n-k} + \frac{1}{k} \right] \\
 &= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{k}{(n-k)k} + \frac{n-k}{k(n-k)} \right] \\
 &= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{n}{(n-k)k} \right] \\
 &= \frac{(n)!}{(k)!(n-k)!} \\
 &= \binom{n}{k}
 \end{aligned}$$

Note: The key algebra here is that $x \cdot [(x-1)!] = x!$ for all positive integers x .

Symmetry of Combinations via Combinatorial Proof

Inspecting Pascal's Triangle, it's fairly easy to see that

$$\binom{n}{k} = \binom{n}{n-k}$$

We prove this combinatorially as follows:

Consider any arbitrary set of k items out of n . Its selection can be matched with the $n-k$ items NOT SELECTED. This is a one-to-one correspondence between the two sets, so the set sizes must be equal. Namely, for each combination of k items out of n , I can determine the unique $n-k$ items not selected. Here is an illustration of this identity using our candy example with $n = 5$, $k = 3$:

<u>In Set</u>		<u>Not In Set</u>
{S,O,H}	→	{M,R}
{O,H,M}	→	{S,R}
{S,O,M}	→	{H,R}
{O,H,R}	→	{S,M}
{S,O,R}	→	{M,H}
{O,M,R}	→	{S,H}
{S,H,M}	→	{O,R}
{H,M,R}	→	{S,O}
{S,H,R}	→	{O,M}
{S,M,R}	→	{O,H}

This can be proven algebraically also and is left as an exercise.

Another Beautiful Combinatorial Proof

Here is another really nice identity that has a much easier combinatorial proof than algebraic one:

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

The RHS represents choosing n items out of $2n$ items. Now, let's split up our counting in a different way.

Color n of the objects red and n of the objects blue.

One way to split up our counting is as follows:

- 0) 0 red objects, n blue objects
- 1) 1 red object, $n-1$ blue objects
- 2) 2 red objects, $n-2$ blue objects
- 3) 3 red objects, $n-3$ blue objects
- ...
- $n-1$) $n-1$ red objects, 1 blue object
- n) n red objects 0 blue objects

Let a row on this chart be row i . Then that row reads that we should choose i red objects and $n-i$ blue objects. We can do this in $\binom{n}{i} \binom{n}{n-i}$ ways. But, using the symmetry identity proven on the last page, this is equal to $\binom{n}{i} \binom{n}{i} = \binom{n}{i}^2$. Finally to add up all of our categories, we just have to sum up this expression as i ranges from 0 to n inclusive. Hence, when adding up the number of ways to choose n objects out of $2n$ objects split into these $n+1$ categories, we get:

$$\binom{2n}{n} = \sum_{i=0}^n \binom{n}{i}^2$$

as desired!

Ant Moving Problem

Let an ant start at $(0, 0)$ on a Cartesian grid. In a single move the ant can move one unit in the positive x-axis or one unit in the positive y axis. Let's say the ant wants to go to the location (a, b) . How many ways can the ant do it?

Ultimately, the ant's path can be viewed as a series of $a+b$ moves, where each move is either U (for up) or R (for right). So, for example, if the target location was $(6, 3)$, a valid set of moves to get there would be RUURURRRR. There is a one to one correspondence between strings of 6 Rs and 3Us and all possible paths the ant could take. The number of such strings is $\frac{9!}{6!3!}$, using the permutation formula.

- Alternatively, we can look at this problem knowing that the ant makes 9 moves, and the ant must choose 6 of those 9 moves to be right. The ant can do this in $\binom{9}{6}$ ways. Of course, equally valid, the ant could just choose 3 out of the 9 ways to move U, which can be done in $\binom{9}{3}$ ways.

All three of these answers are correct, and each one represents a slightly different way of viewing the items that are being counted. (Of course, numerically, these three expressions equal the same value, 84.)

Required Stopping Point

Let's say that the ant wants to go to (10, 12) but needs to stop at the intersection (6, 3) on the way to his final destination.

Well, the ant just has to get to (6, 3), which can be done in $\binom{9}{6}$ ways.

From there, the ant has to move 4 units to the right and 9 units up, which can be done in $\binom{4+9}{4}$ ways.

Since any of the first paths can be paired with any of the second paths, we are counting ordered pairs and need to multiply. The total number of paths is $\binom{9}{6} \binom{13}{4}$. (Note: You can probably do some fun fraction reduction here since the 9! will completely cancel!)

Forbidden Location

Now let's say instead of having to stop at (6, 3), our ant wants to avoid (6, 3). We can use subtraction principle.

The total number of paths is $\binom{22}{10}$. Of these, we already established that $\binom{9}{6} \binom{13}{4}$ go through (6, 3), so we just subtract these out of the total to get the final answer: $\binom{22}{10} - \binom{9}{6} \binom{13}{4}$.

Open Question: How might we deal with 2 forbidden locations?

Ascending Strings

An ascending string is one where all of the vowels and consonants appear in alphabetical order, relatively speaking. Thus, “GAMERS” is an ascending string since the consonants, G, M, R and S, are already in alphabetical order, relative to their position in the string, and the vowels, A and E are also in alphabetical order, relative to one another. How many ascending strings can be made out of the letters A, B, C, D, E, F, G, H, I, O, and U?

First of all, we must simply place the location of both the vowel and consonants in the string. Out of the 11 locations for all of the letters, 5 must be chosen for the vowels. This can be done in $\binom{11}{5}$ ways.

Once these positions are fixed, then we must arrange the vowels. BUT, we realize that vowels, once their locations have been chosen, are FORCED to go in alphabetical order. Thus, there is ONLY one valid arrangement for the vowels once their locations are chosen. Similarly, once the consonant locations are chosen, there is ONLY one valid arrangement of those also – in alphabetical order. Thus, once the locations of the vowels are chosen, the rest of the ascending string falls into place. Thus, the number of valid ascending strings is $\binom{11}{5}$.

Forbidden Locations

You must arrange 3 oak trees, 4 maple trees and 5 birch trees in a line such that no two of the birch trees are adjacent to one another. In how many ways can you arrange the trees? (Assume that two oak trees are indistinguishable, as are two maples and two birches.)

It is tempting to pick four trees to be in between the five birch trees, but that approach leads to confusing situations.

Instead, it is better to take the following approach:

Place the 7 non-birch trees (which will be designated with an N below):

__ N __ N __ N __ N __ N __ N __ N __

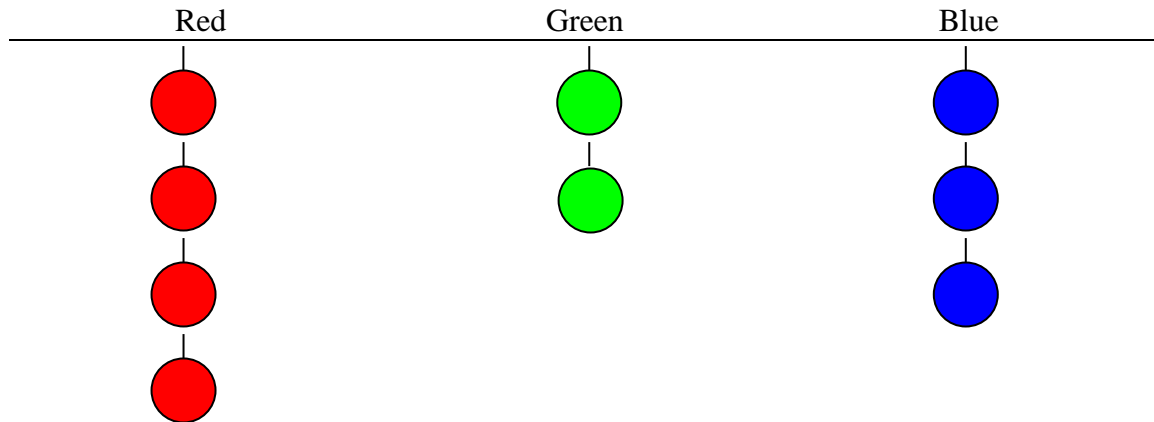
Notice that there are 8 slots in which the birch trees can be placed. We can choose any five of those slots for the birch trees, which can be done in $\binom{8}{5}$ ways. Then, we can arrange the non-birch trees in the number of ways we can permute 3 oaks and 4 maples, which is just $\frac{7!}{3!4!} = \binom{7}{3}$. (Alternatively, we are choosing 3 slots out of the seven Ns to place the Oak trees.)

Thus, the final answer, since each of these choices is independent and combined together for one arrangement is their product,

$$\binom{8}{5} \binom{7}{3}$$

Shooting Targets

If you must shoot the bottom-most target available when shooting at a particular color, in how many ways (orders) can you shoot all of the targets?



The key here is to realize that if you are shooting at a target of a particular color, you are forced to hit one of them. Thus, a string of 4 Rs (standing for red), 2 Gs (standing for green) and 3 Bs (standing for blue) corresponds to a single order of hitting the targets. Each different string corresponds to a different way to hit the targets. For example, the string RGG BRRBBR corresponds to hitting the 4th red, the 2nd green, the 1st green, the 3rd blue, the 3rd red, the 2nd red, the 2nd blue, the 1st blue and the 1st red, in that order.

Thus, the total number of ways to hit the targets down is the number of permutations of 4 Rs, 2 Gs and 3Bs, which is $\frac{9!}{4!2!3!} = 1260$. We can view as a combination as well: first we select 4 slots out of 9 to place the Rs. Then we select 2 slots of the remaining 5 to place the Gs: $\binom{9}{4} \binom{5}{2} = 126 \times 10 = 1260$.

Proof of the Binomial Theorem

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}$$

$$(x + y)(x + y)(x + y)(x + y)(x + y)$$

Each term is going to have some x's and some y's. The highlighted blue terms yield the product x^3y^2 . But another time we get that same term is with these selections:

$$(x + y)(x + y)(x + y)(x + y)(x + y)$$

So the question is, how many times does the term x^3y^2 appear in the expansion of multiplying out $(x+y)^n$?

Basically, we are choosing 3 of the terms out of the 5 terms to get our x item, leaving the y item to be selected from the other 2 terms. But, by definition, we can make this selection in $\binom{5}{3}$ ways.

More generally in the expansion of $(x+y)^n$, the term $x^i y^{n-i}$ appears exactly $\binom{n}{i}$ times.

$$(2x - 3y)^5 =$$

$$(2x)^5 + 5(2x)^4(-3y)^1 + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)^1(-3y)^4 + (-3y)^5 =$$

$$32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5.$$