Counting

Although counting seems to be a simple straightforward task, many counting questions end up being quite counterintuitive.

A couple fairly intuitive rules, the sum and product rules, will first be introduced. Using these, we will derive some fairly interesting results and show how to tackle more difficult counting problems. One of the things to keep in mind here is that depending on HOW you view a counting problem, it could be quite easy or extremely difficult. One should not be discouraged if their initial approach is foiled or seems too difficult. Often times, different approaches to the same problem will eventually yield a fairly simple solution.

Sum Rule

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be done in m+n ways.

This is the book definition. Since most counting questions do not involve any action, I prefer the following version of the definition:

If you have m objects in one group, and n objects in a second group, which is completely separate from the first, (meaning that they share no common objects), then the total number of objects in both groups is m+n.

Here is a set definition: *The cardinality of the union of two disjoint sets is the sum of each set's cardinality.*

Here is a simple example problem using the sum rule:

If all of Richard's beer bottles are either made by Budweiser or Miller, and he has 48 Budweiser bottles and 24 Miller bottles, how many bottles of beer does Richard have all together?

Obviously, 48 + 24 = 72, or 3 cases worth :)

When applying the sum rule, it is important to do the following two things:

1) Count every object in question
2) Do not double count any object

In this problem, it was fairly easy to maintain these two "rules". We were told that Richard only has 2 types of beer bottles, thus we knew we were not forgetting to count any of them. Also, it is clear that a Budweiser bottle can not also be a Miller bottle. You will find that in other problems it is more difficult to count using the rules listed above.

Product Rule

If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage, and n possible outcomes for the second stage, then the total procedure can be carried out in mn ways.

Here's a definition using sets: *The cardinality of the cartesian product of two sets is the product of each set's cardinality.*

The applicability of this rule hinges upon the independence of the number of outcomes for each of the first and second stages. For example, consider the number of ways of picking at least one ace from a deck of cards when drawing a pair of cards. If you happen to pick an ace first, then there are a possible 51 cards you can pick for your second choice. BUT, if you do not pick an ace first, you must pick one of four cards instead. In any event, what we see is that the number of possible ways to execute the second choice is NOT always the same. Thus, we can NOT apply the produce rule in such a simple manner.

However, here is an example where the product rule is applicable:

A class is going to elect one male and one female representative. There are 12 boys in the class and 13 girls. How many possible pairs of representatives can the class elect?

Now, we can use the product rule to get 12x13 = 156 possible pairs. No matter which boy gets elected to represent the class, there will STILL BE 13 choices for the female representative. To visualize the answer, we can enumerate all possible pairs using a 12x13 table.

Let's go through a problem that utilizes both of these rules:

Here are the rules for a valid phone number:

1) The first three digits must all be different.
2) The first digit can not be a 1.
3) If the first digit is a 5, the last four digits must all be the same.

Based on our rules, we must group our phone numbers into two separate groups:

A: All numbers that start with 5
B: All numbers that do NOT start with 5

First, we will count all the phone numbers that fit in group A.

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \textbf{-}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$$
$$\ \ 1\quad 9\quad 8\qquad 10\ \ 1\quad 1\quad 1$$

For the first number, we have 1 choice, 5.
For the second number, we have 9 choices, everything but 5.
For the third number, we have 8 choices.
For the fourth number we have all 10 choices, but for the rest we only have 1 choice, based on our rules.

Thus, we have 1x9x8x10x1x1x1 = 720 phone numbers that start with 5.

Now, let's count the phone numbers that do not start with 5:

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \textbf{-}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ ,$$
$$\ 8\quad 9\quad 8\qquad 10\ \ 10\ \ 10\ \ 10$$

for a total of 8x9x8x10x10x10x10 = 5760000 phone numbers that do not start with 5.

Using the sum rule, we add this to our other value to get a total of 5,760,720 total possible phone numbers under the given rules.

Here is a quick problem for you: How many 3 letter words can you form that have two consonants and a vowel?

Subtraction Rule

In Rainman, there's a scene where Dustin Hoffman's character counts the number of toothpicks on the floor that have just fallen out of a big box of toothpicks. Almost immediately, he says, "248, definitely 248." Since he was a savant, he was able to actually count all of the toothpicks on the floor. But for the rest of us, who count at a normal pace, verifying that would take a few minutes. But instead, the onlookers in the movie were instantaneously able to verify his count.

The toothpicks all came from a previously unopened box of 250 toothpicks. When the box was examined right after the toothpicks fell out, 2 were found. So this is the subtraction rule:

The number of elements in the set ¬A from an universe U is simply |U| - |A|.

Here's a short problem that utilizes the subtraction principle:

How many seven digit numbers (where leading 0s are acceptable) do not contain the substring 1234?

Using the product rule, there are 10^7 valid phone numbers.

Of these, we need to count how many contain the substring 1234. Here are our possibilities:

1 2 3 4 __ __ __   __ 1 2 3 4 __ __ __
__ __ 1 2 3 4 __   __ __ __ 1 2 3 4

Each of account for 10^3 numbers with the substring 1234. Thus, our final answer is $10^7 - 4(10^3) = 9996000$.

Division Rule

On occasion, we will accidentally count each item we want to count multiple times. If we are lucky enough to count each item the same number of times, then we can just get our answer by dividing. So the division rule is as follows:

If we are trying to count some items, and we count each item exactly k times, then we can get the desired count by taking our count and dividing by k.

Here is a small example of the division rule:

Let's say at a party where there were five people (call them A, B, C, D and E), we know how many hands each person shook:

A = 2
B = 2
C = 4
D = 3
E = 1

What if we wanted to know the total number of handshakes that occurred at the party?

If we add up the five numbers above: 2 + 2 + 4 + 3 + 1 = 12, we get the total number of handshake events from the view of each person added up. But, recall that each handshake event occurred between 2 people, so for each handshake, a total of 2 will be contributed to the count of 12 we have. Thus, when we counted 12, we counted each handshake precisely twice, once for each person shaking hands. Thus, the total number of handshakes that occurred is 12/2 = 6.

For this set of data, we can even determine who shook hands:

Firstly, C shook everyone's hand, so we have:

CA, CB, CD, CE

Of those who remain, E shook no one else's hand and D must have shaken two other hands so we add to the list:

DA, DB.

Now, we have the full set of handshakes that occurred!

Incidentally, this idea, that if ask everyone at a party how many hands they shook and add up that number, we must get an even number is known as the Handshake Lemma!

In graph theory (you will see this in Computer Science II), one way to think about a graph in terms of a party is that each person is a dot (vertex) and for each handshake we draw a line connecting the appropriate dots (edges). The degree of a vertex is the number of edges that are adjacent to that edge. If we sum the degrees of all the vertices in a graph, we get two times the number of edges. This is the Handshake Lemma with respect to graph theory: $\sum_{v \in V} deg(v) = 2|E|$, where V is the set of vertices of the graph and E is the set of edges.

Currently, for our purposes, we will use the division rule to derive formulas for both permutations and combinations.

Inclusion-Exclusion Principle

This was covered when we learned about sets, but just in case the sets we are dealing with in a counting problem aren't disjoint, we can "fix" our error by subtracting out what was counted twice. So, we have:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

So, this is really a fix to the Addition Rule. Recall, for three sets we have:

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C$$

Permutations

A permutation of objects is simply an arrangement of those objects. A common example of permutations deals with words. How many "words" can you form out of the letters A, B, C, and D, without repeating a letter. Essentially, the question boils down to how many ways can you order a set of objects. In this case, we know we have four choices for the first object. Once we have made that choice, it is clear that we will always have three choices for the next object. In particular, we find the following:

$$\underline{\quad} \; \underline{\quad} \; \underline{\quad} \; \underline{\quad}$$
4 x 3 x 2 x 1   = 24 possible permutations.

In general, imagine the number of ways to permute n objects. Using the same process as above we find:

$$\underline{\quad} \; \underline{\quad} \quad\;\; \underline{\quad} \; \underline{\quad}$$
n   x(n-1)x ... 2 x 1   = n! possible permutations.

(Note: For those of you unfamiliar with the ! notation, it is read as "factorial". n! is simply defined as the product of all the positive integers in between 1 and n, inclusive.)

Now, consider the following problem:

Given n distinct objects, how many permutations are there of r of those objects, where $1 \leq r \leq n$.

Using the product rule as before, we find:

$$\underline{\quad n \quad} \times \underline{(n-1)} \times \ldots \quad \underline{(n-k+1)} = n!/(n-k)! \text{ possible permutations.}$$

Now, consider a situation where you are permuting objects, where some of those objects are indiscernible. For example, how many permutations are there of the word BEER? At first we would say 24, but the problem is that we can not tell the difference between the two E's. To look at the problem more closely, imagine giving subscripts to the 2 Es to discern them, and THEN listing out the permutations as follows:

BEER	B E_1 E_2 R	B E_2 E_1 R
BERE	B E_1 R E_2	B E_2 R E_1
BREE	B R E_1 E_2	B R E_2 E_1
EBER	E_1 B E_2 R	E_2 B E_1 R
EBRE	E_1 B R E_2	E_2 B R E_1
EEBR	E_1 E_2 B R	E_2 E_1 B R
EERB	E_1 E_2 R B	E_2 E_1 R B
ERBE	E_1 R B E_2	E_2 R B E_1
EREB	E_1 R E_2 B	E_2 R E_1 B
RBEE	R B E_1 E_2	R B E_2 E_1
REBE	R E_1 B E_2	R E_2 B E_1
REEB	R E_1 E_2 B	R E_2 E_1 B

Thus, we see that there are really 12 distinct permutations of BEER. (12 pack, 12 permutations, coincidence? Hmmm....)

In particular, for every distinct permutation of BEER there are 2 "possibilities" listed on the RHS with all the permutations, assuming distinct E's. In fact, we can see that if we had a word with 3 E's, there would be six "possibilities listed, 1 for each permutation of the 3 E's, assuming that they are distinct.

In general, the number of permutations of n letters, where the number of repetitions of each letter are n_1, n_2, n_3, ... n_k, is

$n!/(n_1! \, n_2! \, ...n_k!)$

The reasoning behind this is the following:

If we assume that each of the n letters is distinct (ie. if all the similar letters had subscripts to distinguish them) then n! is the correct answer. But, as we saw in our example before, using this method of counting, we count the same "word" multiple times. Our goal is to figure out HOW many times we count each word.

If letter 1 appears n_1 times, then we can rearrange each of those n_1 characters in exactly $n_1!$ ways, keeping our original permutation unchanged. For each of those $n_1!$ arrangements, we have the following:

If letter 2 appears n_2 times, then we can rearrange each of those n_2 characters in exactly $n_2!$ ways, keeping our original permutation unchanged.Since these are independent "events" so to speak, we use the product rule to calculate the total number of times we have recounted the same permutation. The final answer ends up becoming the product of all the frequencies of all the letters, or more mathematically, $n_1! \, n_2! \, ...n_k!$

Thus, to get the true number of permutations with repeats, you must divide n! by the value above.

Combinations

Now, consider the following two questions:

1) Given a set of n objects, how many total possible subsets can you have of those objects? A subset simply is any group (with a size in between 0 and n, inclusive) of the n objects. The order in which the members of the subset are picked does not matter.

2) Given a set of n objects, how many subsets of size k can you have of those objects.

For the first question, we can view creating a subset of a group of n objects in the following manner:

First, decide whether or not element #1 is in the subset.
Next, decide whether or not element #2 is in the subset.
.
.
.

Each of these choices is independent from each of the other choices, and for each choice we have two options. Thus, we can use the product rule to count the total number of possible subsets:

$$\underline{\quad}, \underline{\quad}, \underline{\quad}, \cdots \underline{\quad}$$
$$2 \text{ x } 2 \text{ x } 2 \text{ x} \ldots 2 = 2^n \text{ total subsets of a set of size n.}$$

Now, let's consider the second question. This seems to be similar to counting permutations, with one key difference: the ORDER in which the elements are picked does NOT matter.

Thus, when we get $n!/(n-k)!$ permutations of k objects out of n objects, we see that we are over counting. In particular, each combination of items is counted k! times. Here is an example to illustrate this fact:

Consider the combinations of size 3 of the letters K, N, I, G, H, T, and S. One of these combinations is G, I, and N. Now, consider listing all of the permutations of size 3 that "correspond" to this one combination:

GIN, GNI, IGN, ING, NGI, and NIG. In particular, there are 3! of these combinations. Thus, when we count the number of permutations of size 3, we are counting each combination 3! times. In general, when we count the number of permutations of size k, we are counting each combination k! times.

So, the answer to our original question is $n!/((n-k)!k!)$.

This value comes up so often in counting and other parts of discrete math that there are special symbols for it. The one I will use in my notes from now on is the following:

$_nC_k = n!/((n-k)!k!)$. This can be read as "n choose k".

Consider the following problem which involves combinations:

Consider choosing a senate committee of five members. There are 35 women in the senate and 65 men. How many committees can you pick with exactly three men?

We know that if we have three men, we must have exactly two women on the committee. The choices of the men and women are independent. We can choose the women in $_{35}C_2$ ways, and we can choose the men in $_{65}C_3$ ways. The total number of committee choices will simply be the product of these two values.

Now, consider that we want to know the total number of committees we can pick if we must have at least 1 women on the committee.

There are two approaches, we can separate our counting into 4 mutually exclusive sets:

Committees with exactly 1 woman, committees with exactly 2 women, committees with exactly 3 women, committees with exactly 4 women and committees with exactly 5 women.

However, this seems quite tedious. The more simple idea would be to count the total number of committees with no women, and subtract this from the total number of committees:

committees with no women = $_{65}C_5$, since 5 men must be chosen
total # of committees = $_{100}C_5$.

Thus, the number of committees with at least one woman is:

$_{100}C_5$ - $_{65}C_5$.

Finally, here is a problem for you all: On an exam you must answer 7 of 10 questions. How many different combinations of questions can you answer with the following restriction:

You must answer AT LEAST 4 of the first 6 questions.