Random Algebra Notes

There are many types of algebra problems and there are a few general tips for the whole set of these sorts of problems:

1) Clearly read the problem and identify unknown piece(s) of information. For each unknown piece that you feel might be relevant, assign a variable name. For example, "let A equal annie's age right now," or "let L equal the length of the small rectangle."

2) Aim to create as few variables as possible. (For example, if you know that y is twice as big as x, then just use x and 2x from the start.

3) Write equations based on the information given in the problem based on the variables you created and other values given in the problem.

4) Figure out how to use standard algebra techniques to combine the equations to help solve for the value of the relevant variables.

5) On occasion, you can answer the posed question without every figuring out the value of each individual variable you created. For example, if you are asked to find x+y, you may never actually have to find either x or y. Thus, be on the lookout, given your equations, how you might be able to isolate the desired expression, and maybe you can get its value without ever getting the values of the individual components.

The remainder of these notes will split algebra problems into "categories" based on domain. Since a separate set of notes has been devoted to distance = rate times time problems, those won't be covered in these notes.

"Standard Problems" with two linear variables

There are many problems where the variables are clearly delineated in the problem, there are two variables and two equations. Here is an example:

Jasmine has 20 coins, all of which are nickels or quarters. The total value of her 20 coins is $3.60. How many nickels does she have?

Let n be the number of nickels and q be the number of quarters. Then we have the two equations:

$n + q = 20$
$5n + 25q = 360$

Substitute $q = 20 – n$:

$5n + 25(20 – n) = 360$
$5n + 500 – 25n = 360$
$20n = 140$
$n = 7$, so there are 7 nickels.

A couple pieces of commentary on this specific problem (which won't always apply to these sorts of problems with two clear variables and equations):

You can set the number of quarters to 20 – n from the beginning, so that your problem only has one variable.

Also, it's easy to see that the maximum money she can have is $5.00, because that's the value of 20 quarters. For each quarter she exchanges with a nickel, she loses 20 cents. Thus, since her "loss" from the maximum she could have is $5 - $3.60 = $1.40, she made this exchange for a nickel, 7 times.

Age Problems

In these problems, usually you are given information about a person (or two people's) age(s) at two different times. Perhaps something like:

Fifteen years ago, Sally was three times as old as her daughter Sandra. Today, if you multiply Sandra's age by two and subtract seven, you get Sally's age. How old is Sally today?

In these problems, since there are different times involved, when you create a variable for someone's age, it's important to specify at what time that variable represents. Here is a solution to this problem, keeping this in mind:

Let x be Sandra's age, 15 years ago. This means that Sally was 3x years old 15 years ago.
Thus, today, Sandra is x + 15 years old and Sally is 3x + 15 years old. Finally, using the given information, we find that:

$2(x + 15) - 7 = 3x + 15$
$2x + 30 - 7 = 3x + 15$
$8 = x$ → so, 15 years ago they were 8 and 24. Today, they are 23 and 39. *So Sally is 39 today.*
Thus, depending on when we are talking about, we have two different ages for each person. When setting up equations, it's important to keep straight each detail about the equation (is this equation about today or 15 years ago?).

Fraction Decomposition for Algebraic Simplification

In some problems, there are fractions with monomials in both the numerator and denominator. Often times, the expression can be simplified by noticing that the numerator is (almost) a multiple of the denominator. More formally, polynomial division can simplify fractions of polynomials. Here is an example:

Find an expression for y in terms of x when x and y satisfy the following equation and $y \neq -2$ and $x \neq 0$:

$$\frac{2y+5}{y+2} = \frac{x-1}{x}$$

$$\frac{2(y+2)+1}{y+2} = 1 - \frac{1}{x}$$

$$2 + \frac{1}{y+2} = 1 - \frac{1}{x}$$

$$\frac{1}{y+2} = -1 - \frac{1}{x}$$

$$\frac{1}{y+2} = -\frac{(x+1)}{x}$$

$$y + 2 = -\frac{x}{x+1}$$

$$\boldsymbol{y = -2 - \frac{x}{x+1}}$$

Work Problems
These are problems where it takes a worker X units of time to complete a task working alone, but then we can combine workers to make the task go faster. For example, a question could be, if one worker takes X hours to complete a task working alone, and another worker takes Y hours to complete the same task working alone, how long will they take working on the task together? These problems always assume that combining workers scales. Namely, if more than one person is on the job, each person still gets as much done (per unit of time) as they would if they were working by themselves. (Clearly, in real life this isn't true.)

If it takes X units of time to complete the task for the first person, then in one unit of time, that person completes $\frac{1}{X}$ of the task. Similarly, the other person, in one unit of time, completes $\frac{1}{Y}$ of the task. Let t be the amount of time (in units) that they complete the task together. Then, we have:

$$\frac{t}{X} + \frac{t}{Y} = 1$$

$$t(\frac{Y + X}{XY}) = 1$$

$$t = \frac{XY}{X + Y}$$

There are variations on work problems, but the key is usually creating expressions for how much of a task each individual worker gets done in one unit of time, and then using the given information to set up corresponding equations. These equations tend to have fractions, so the most difficult part often deals with the actual algebra manipulating these expressions in equations.

Clock Problems
The key to clock problems is that the minute hand travels 360 degrees in one hour while the hour hand travels 360/12 = 30 degrees in one hour. If we convert these rates to minutes, then we see that the minute hand rotates 6 degrees every minute and the hour hand rotates ½ a degree each minute. Hands that are exactly opposite of each other create a degree measure of 180 degrees. Here is a sample problem: Sometime in between 2 and 3 pm, the minute hand is exactly opposite the hour hand. How many minutes after 2 pm does this occur? Leave your answer as a fraction in lowest terms. We can let x equal the number of minutes past 2 pm this occurs. After x minutes, the minute hand is at degree measure 6x. After x minutes, the hour hand is at degree measure 60 + x/2, because it already started at degree measure 60 (measured clockwise from the top). We want the latter angle to be 180 less than the former angle (meaning, if we add 180 to it, it'll equal the other angle), which gives us:

$$6x = 60 + \frac{x}{2} + 180$$

$$\frac{11x}{2} = 240$$

$$x = \frac{480}{11}$$

Digit Problems
In some problems interchanging digits is discussed, or the digits of a number are used. In these problems, it makes sense to assign variables to each digit, while recognizing that those variables can only equal integers 0 through 9. (Sometimes, 0 might not be allowed.)

Here is a problem that shows a property of exchanging digits.

Both Tristan and Sammy are in between 10 and 99 years old. If you exchange the tens and ones digit of Tristan's age, you get Sammy's age. Prove that the difference of their ages is divisible by 9.

Let Tristan's age be 10a + b, where a is the tens digit of her age and b is the ones digit of her age.

Then, Sammy's age is 10b + a.

Subtract these: $10a + b - (10b + a) = 10a + b - 10b - a = 9a - 9b = 9(a - b)$

Because a and b are integers (digits specifically), a – b is an integer and this difference is divisible by 9.

Mixture Problems
In mixture problems, usually there are two liquids mixed together at some percentage (say 90% water, 10% acid) and then some quantity of one of the two liquids is added, resulting in a mixture that is a different overall percentage. One standard way to handle these problems is to simply represent or calculate the total number of ounces of the mixture at each snapshot in time that should be one of the two liquids. Here is an example of a mixture problem (from one of my final exams):

For this question, assume that if x_1 ounces of liquid 1 at temperature t_1 is mixed with x_2 ounces of liquid 2 at temperature t_2, the resulting mixture is $x_1 + x_2$ ounces with a temperature of $\frac{t_1x_1+t_2x_2}{x_1+x_2}$. Arup's wife, Anita, likes her tea mixed with milk. (She also likes it with sugar, but I didn't want to make this problem too hard!) Unfortunately, she has a very specific preference for both the ratio of tea to milk and the temperature of the overall mixture. Specifically, she wants a 6:1 ratio of tea to milk at a temperature of 180° F. On Arup's first attempt, he mixed 8 ounces of tea at 200° F with half an ounce of milk at 40° F. It's not possible for Arup to add any more tea to the mixture, but he is allowed to add more milk and he can heat that milk to any temperature greater than 40° F, if necessary, before he adds it. How much milk should Arup add to the tea and at what temperature should he heat it before adding it, to fix the tea?

In this problem, we see that the current mixture has 8 ounces of tea and ½ ounce of milk. We must add x ounces of milk so that the ratio of the two liquids is 6:1. This gives us the equation:

$$\frac{8}{\frac{1}{2}+x} = \frac{6}{1}$$

Solving for x, we get $6\left(\frac{1}{2}+x\right) = 8 \rightarrow 3 + 6x = 8 \rightarrow 6x = 5 \rightarrow x = \frac{5}{6}$

Notice that once we know this information, it really simplifies the problem because the quantity of milk to add to the mixture is no longer a variable. Now, the only variable is the temperature of this milk. From this point, there are two strategies:

1) Calculate the temperature of the old mixture via the given formula, and then use this as a single substance of uniform temperature that will be mixed with milk at a new temperature (unknown).

2) Look at how the formula is designed and redesign it for 3 liquids, which would be the original 8 ounces of tea, original ½ ounce of milk, and the last 5/6 ounces of milk, at temperatures 200, 40 and t, respectively.

For these notes, we'll solve this both ways, and hopefully you can see the equivalence of the methods.

Strategy 1

Calculate temperature of 8.5 ounce mixture: $\frac{8(200)+.5(40)}{8.5} = \frac{1620}{8.5} = \frac{3240}{17}$.

Now, let t be temperature of new milk, so we have:

$$\frac{8.5 \times \frac{3240}{17} + \frac{5}{6}t}{8.5 + \frac{5}{6}} = 180$$

$$\frac{3240}{2} + \frac{5}{6}t = 180 \times (\frac{28}{3})$$

$$1620 + \frac{5t}{6} = 1680$$

$$\frac{5t}{6} = 60, so\ t = \frac{360}{5} = 72$$

Strategy 2

We can generalize the formula for n liquids just by multiplying each liquid's temperature by its weight in the numerator, dividing by the total weight in the denominator. This gives us:

$$\frac{8(200) + .5(40) + \frac{5}{6}t}{\frac{28}{3}} = 180$$

$$1620 + \frac{5}{6}t = 1680$$

From this point on, the solution is the same as strategy 1.