Creating Algorithms

Now that we've discussed analyzing algorithms, let's briefly discuss designing efficient algorithms. The latter is much more difficult to do than the former because it requires more creativity. (Coming up with a completely new algorithm to solve a problem faster than another algorithm is more difficult than counting the number of simple steps in an algorithm that someone else has already created for you.)

In this lecture we will analyze two separate problems. For each problem we will look and three solutions, each of which is an improvement on the previous one. In this manner, we will see how it might be possible to first find a solution to a problem, and then later "fine-tune" it to be more efficient with respect to run-time.

Sorted List Matching Problem

Given two sorted lists of names, output the names common to both lists.

Perhaps the standard way to attack this problem is the following:

For each name on list #1, do the following:

a) Search for the current name in list #2.

b) If the name is found, output it.

If a list is unsorted, steps a and b may take O(n) time. Can you tell me why?

BUT, we know that both lists are already sorted. Thus we can use a binary search in step a. From CS1, we learned that this takes O(log n) time, where n is the total number of names in the list. For the moment, if we assume that both lists are of equal size, then we can safely say that the size of list #2 is about ½ the total input size, so technically, our search would take O(log n/2) time, where n is the TOTAL SIZE of our input to the problem. Using our log rules however, we find that $\log_2 n = (\log_2 n/2) + 1$. Thus, it's fairly safe to assume for large n that our running time is simply $O(\log_2 n)$.

Now, that is simply the running time for 1 loop iteration. But how many loop iterations are there? (Assume that there are n/2 names on each list, again, where n is the TOTAL SIZE of the input.) Under our assumption, there will be n/2 loop iterations, so our total running time would be $O(n \log_2 n)$. Why did I not divide the expression in the Big-O by 2?

A natural question becomes: Can we do better? The answer is yes. What is one piece of information we have that our first algorithm does NOT assume?

That list #1 is sorted. You'll notice that our previous algorithm will work regardless of the order of the names in list #1. But, we KNOW that this list is sorted also. Can we exploit this fact so that we don't have to do a full binary search for each name?

Consider how you'd probably do this task in real life...

List #1	List #2
Adams	Boston
Bell	Davis
Davis	Duncan
Harding	Francis
Jenkins	Gamble
Lincoln	Harding
Simpson	Mason
Zoeller	Simpson

You'd read that Adams and Boston are the first names on the list. Immediately you'd know that Adams wasn't a match, and neither would any name on the list #1 alphabetically before Boston. So, you'd read Bell and go on to Davis. At this point you'd deduce that Boston wasn't on the list either, so you'd read the next name on list #2 – voila!!! A match! You'd output this name and simply repeat the same idea. In particular, what we see here is that you ONLY go forward on your list of names. And for every "step" so to speak, you will read a new name off one of the two lists. Here is a more formalized version of the algorithm:

1) Start two "markers", one for each list, at the beginning of both lists.
2) Repeat the following steps until one marker has reached the end of its list.
a) Compare the two names that the markers are pointing at.
b) If they are equal, output the name and advance BOTH markers one spot.
If they are NOT equal, simply advance the marker pointing to the name that comes earlier alphabetically one spot.

<u>Algorithm Run-Time Analysis</u>
For each loop iteration, we advance at least one marker.
The maximum number of iterations then, would be the total number of names on both lists, which is n, using our previous interpretation.

For each iteration, we are doing a constant amount of work. (Essentially a comparison, and/or outputting a name.)

Thus, our algorithm runs in O(n) time – an improvement over our previous algorithm.

A final question one must ask is, can we solve this question in even less time? If yes, what is such an algorithm, if no, how can we prove it?

Our proof goes along these lines: In order to have an accurate list, we must read every name on one of the two lists. If we skip names on BOTH lists, we can NOT deduce whether we would have matches between those names or not. In order to simply "read" all the names on one list, we would take O(n/2) time. But, in order notation, this is still O(n), the running time of our second algorithm. Thus, we know we can not do better in terms of time, (within a constant factor), of our second algorithm.

Sample Algorithm Development and Analysis

The Prefix Average problem is as follows:

Given an array of values, X[0..n-1], compute a second array with A with intermediate averages such that A[i] is the average of X[0], X[1], … X[i].

The straightforward algorithm is as follows:

For each array element A[i]:
Compute this value by adding X[0], X[1], .., X[i] in a loop, then dividing by i.

In code we have:

```
public static int[] prefixave(int [] X) {
    int [] A = new int[X.length];

    // Loop to successively compute each average.
    for (int i=0; i<A.length; i++) {
        A[i] = 0;
        for (int j=0; j<=i; j++)  // Sum X[0] to X[i].
            A[i] += X[j];
        A[i] = A[i]/(i+1); // Compute average from sum.
    }
    return A;
}
```

Hopefully it is evident that that this algorithm will work. The question is, how long will it take? Notice that the statements A[i]=0 and A[i]=A[i]/(i+1) both execute exactly n times.

The only question is how many times does A[i] += X[j] execute? When i=0, it executes once, with i=1, it executes twice, …, finally when i=n-1, (where n is the length of the array), it executes n times. Thus, the number of times this statement is executed is $1+2+3+\ldots+n = n(n+1)/2 = O(n^2)$.

So, using all of this, the total number of simple statements is $O(n) + O(n^2)$. (The O(n) is for the 2 other loop statements and the for the outer for loop increment statement and comparison.) Using the Big-Oh rules, we find that this algorithm is $O(n^2)$.

But, it seems as if we are doing too much work here. Can we streamline this algorithm? Consider computing just all of the prefix sums first, instead of the averages. We can compute each running sum in a accumulator variable. Then we can simply assign each array element A by dividing the running sum by the number of terms added. From CS1, the efficient way to run an accumulator variable is to initialize it to 0 and then simply add subsequent terms from the array X into the accumulator variable:

```
public static int[] prefixave2(int [] X) {
    int [] A = new int[X.length];
    int s = 0;
    // Loop to successively compute each average.
    for (int i=0; i<ave.length; i++) {
        s += X[i];
        A[i] = s/(i+1); // Compute average from sum.
    }
    return A;
}
```

Why is this an O(n) algorithm?

Maximal Contiguous Subsequent Sum Problem

Maximum Contiguous Subsequence Sum: **given (a possibly negative) integers A_1, A_2, …, A_N, find (and identify the sequence corresponding to) the maximum value of**

$$\sum_{k=i}^{j} A_k$$

For the degenerate case when all of the integers are negative, the maximum contiguous subsequence sum is zero.

Examples:

If input is: {-2, 11, -4, 13, -5, 2}. Then the output is: 20.

If the input is {1, -3, 4, -2, -1, 6}. Then the output is 7.

In the degenerative case, since the sum is defined as zero, the subsequence is an empty string. An empty subsequence is contiguous and clearly, 0 > any negative number, so zero is the maximum contiguous subseqeunce sum.

The O(N^3) Algorithm (brute force method)

```
public static int MCSS(int [] a) {

    int max = 0, sum = 0, start = 0, end = 0;

    // Cycle through all possible values of start and end indexes
    // for the sum.
    for (i = 0; i < a.length; i++) {
        for (j = i; j < a.length; j++) {
            sum = 0;

            // Find sum A[i] to A[j].
            for (k = i; k <= j; k++)
                sum += a[k];
            if (sum > max) {
                max = sum;
                start = i; // Although method doesn't return these
                end = j;  // they can be computed.
            }
        }
    }
    return max;
}
```

General Observation Analysis

Look at the three loops: the *i* loop executes *SIZE* (or N) times. The *j* loop executes *SIZE-1* (or N-1) times. The *k* loop executes *SIZE-1* times in the worst case (when i = 0). This gives a rough estimate that the algorithm is O(N^3).

Precise Analysis Using Big-Oh Notation

In all cases the number of times that, sum += a[k], is executed is equal to the number of ordered triplets (i, j, k) where $1 \leq i \leq k \leq j \leq N^2$ (since *i* runs over the whole index, *j* runs from *i* to the end, and *k* runs from *i* to *j*). Therefore, since *i*, *j*, *k*, can each only assume 1 of n values, we know that the number of triplets must be less than $n(n)(n) = N^3$ but $i \leq k \leq j$ restricts this even further. By combinatorics it can be proven that the number of ordered triplets is n(n+1)(n+2)/6. Therefore, the algorithm is $O(N^3)$.

A Simple Big-Oh Rule

A Big-Oh estimate of the running time is determined by multiplying the size of all the nested loops together. BUT, THERE ARE EXCEPTIONS TO THIS RULE!!!

The O(N^2) Algorithm

Algorithm

```
public static int MCSS(int [] a) {
    int max = 0, sum = 0, start = 0, end = 0;
    // Try all possible values of start and end indexes for the sum.
    for (i = 0; i < a.length; i++) {
        sum = 0;
        for (j = i; j < a.length; j++) {
            sum += a[j]; // No need to re-add all values.
            if (sum > max) {
                max = sum;
                start = i; // Although method doesn't return these
                end = j;  // they can be computed.
            }
        }
    }
    return max;
}
```

Discussion of the technique and analysis

We would like to improve this algorithm to run in time better than O(N^3). To do this we need to remove a loop! The question then becomes, "how do we remove one of the loops?" In general, by looking for uncessary calculations, in this specific case, uncessary calculations are performed in the innerloop. The sum for the subsequence extending from *i* to *j – 1* was just calculated – so calculating the sum of the sequence from *i* to *j* shouldn't take long because all that is required is that you add one more term to the previous sum (i.e., add A_j). However, the cubic algorithm throws away all of this previous information and must recompute the entire sequence!

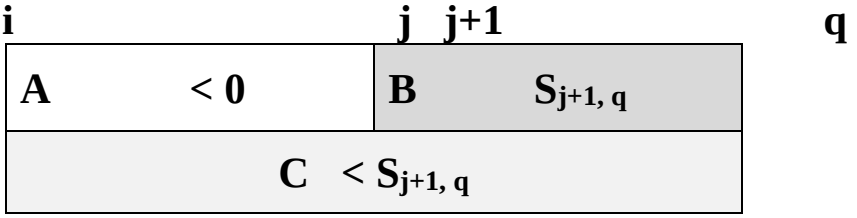
Mathematically, we are utilizing: $\sum_{k=i}^{j} A_k = (\sum_{k=i}^{j-1} A_k) + A_j$ **.**

The O(N) Algorithm (A linear algorithm)

Discussion of the technique and analysis

To further streamline this algorithm from a quadratic one to a linear one will require the removal of yet another loop. Getting rid of another loop will not be as simple as was the first loop removal. The problem with the quadratic algorithm is that it is still an exhaustive search, we've simply reduced the cost of computing the last subsequence down to a constant time (O(1)) compared with the linear time (O(N)) for this calculation in the cubic algorithm. The only way to obtain a subquadratic bound for this algorithm is to narrow the search space by eliminating from consideration a large number of subsequences that cannot possibly affect the maximum value.

How to eliminate subsequences from consideration



If A < 0 then C < B

If $\sum_{k=i}^{j} A_k < 0$ **, and if q > j, then $A_i \ldots A_q$ is not the MCSS!**

Basically if you take the sum from A_i to A_q and get rid of the first terms from A_i to A_j your sum increases!!! Thus, in this situation the sum from A_{j+1} to A_q must be greater than the sum from A_i to A_q. So, no subsequence that starts from index i and ends after index j has to be considered.

So – if we test for sum < 0 and it is – then we can break out of the inner loop. However, this is not sufficient for reducing the running time below quadratic!

Now, using the fact above and one more observation, we can create a O(n) algorithm to solve the problem.

If we start computing sums $\sum_{k=i}^{i} A_k$, $\sum_{k=i}^{i+1} A_k$ **, etc. until we find the first value j such that** $\sum_{k=i}^{j} A_k < 0$ **, then immediately we know that either**

1) The MCSS is contained entirely in between A_i to A_{j-1} OR
2) The MCSS starts before A_i or after A_j.

From this, we can also deduce that unless there exists a subsequence that starts at the beginning that is negative, the MCSS MUST start at the beginning. If it does not start at the beginning, then it MUST start after the point at which the sum from the beginning to a certain point is negative.

So, using this how can we come up with an algorithm?

1) We can compute intermediate sums starting at i=0.
2) When a new value is added, adjust the MCSS accordingly.
3) If the running sum ever drops below 0, we KNOW that if there is a new MCSS than what has already been calculated, it will start AFTER index j, where j is the first time the sum dropped below zero.
4) So now, just start the new running sum from j+1.

Algorithm

```
public static int MCSS(int [] a) {

    int max = 0, sum = 0, start = 0, end = 0, i=0;

    // Cycle through all possible end indexes.
    for (j = 0; j < a.length; j++) {

        sum += a[j]; // No need to re-add all values.
        if (sum > max) {
            max = sum;
            start = i; // Although method doesn't return these
            end = j;  // they can be computed.
        }
        else if (sum < 0) {
             i = j+1; // Only possible MCSSs start with an index >j.
             sum = 0; // Reset running sum.
        }
    }
    return max;
}
```

Discussion of running time analysis

The *j* loop runs N times and the body of the loop contains only constant time operations, therefore the algorithm is O(N).

MCSS Linear Algorithm Clarification

Whenever a subsequence is encountered which has a negative sum – the next subsequence to examine can begin after the end of the subsequence which produced the negative sum. In other words, there is no starting point in that subsequence which will generate a positive sum and thus, they can all be ignored.

To illustrate this, consider the example with the values

5, 7, -3, 1, -11, 8, 12

You'll notice that the sums

5, 5+7, 5+7+(-3) and 5+7+(-3)+1 are positive, but

5+7+(-3)+1+(-11) is negative.

It must be the case that all subsequences that start with a value in between the 5 and -11 and end with the -11 have a negative sum. Consider the following sums:

7+(-3)+1+(-11) (-3)+1+(-11) 1+(-11) (-11)

Notice that if any of these were positive, then the subsequence starting at 5 and ending at -11 would have to be also. (Because all we have done is stripped the initial positive subsequence starting at 5 in the subsequences above.) Since ALL of these are negative, it follows that NOW MCSS could start at any value in between 5 and -11 that has not been computed.

Thus, it is perfectly fine, at this stage, to only consider sequences starting at 8 to compare to the previous maximum sequence of 5, 7, -3, and 1.