Three Dimensional Geometry

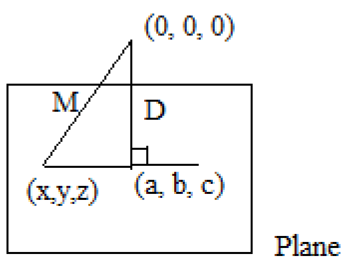
Equations of Planes in Three Dimensions

Normal Vector

In three dimensions, the set of lines perpendicular to a particular vector that go through a fixed point define a plane. To try out this idea, pick out a single point and from this point imagine a vector emanating from it, in any direction. Now, consider the piece of paper that always forms a right angle with this vector. This piece of paper is representing a specific plane in three dimensions. Thus, we need two pieces of information to ascertain the equation of a plane:

1) Its normal vector
2) A point on the plane

Now, consider drawing a vector from the origin (0,0,0) to any point on the plane (x, y, z). Let the normal vector to the plane be ai + bj + ck. The dot product between these two vectors is ax + by + cz. But the dot product between these vectors divide by both of their magnitudes is also the cosine of the included angle. Consider the following picture:



Using the distances labeled in the diagram, we have

$$ax + by + cz = MDcos\theta$$

But, remember that we can get an expression for cos θ by using the right triangle connecting the origin, (a, b, c) and (x, y, z). Thus, we have $cos\theta = \frac{D}{M}$. Thus, we can simplify the equation above to

$$ax + by + cz = D^2$$

Typically, most textbooks will write that the Cartesian form for the equation of the plane is

$$ax + by + cz = D$$

Notice that these two forms are consistent. In the latter form, D no longer represents the distance of the plane from the origin. But, using the former form, we can make the following realization:

$D^2 = a^2 + b^2 + c^2$. Thus, D represents the magnitude of the vector that is the normal to the plane, starting from the origin. If we divide both sides by D, we get the following:

$$\frac{a}{D}x + \frac{b}{D}y + \frac{c}{D}z = D$$

Remember that the coefficients a, b and c are nothing but the components of the normal vector to the plane. In this latter form, we've scaled this normal vector to be a unit vector, a vector of magnitude 1. (This is relatively easy to see because we defined D to be the distance of the point (a, b, c) from the origin.) Thus, this gives rise to the vector equation of a plane:

$$r \cdot \hat{n} = D$$

The position vector (point) r represents an arbitrary point on the plane. (This is equivalent to (x,y,z) in the Cartesian equation.). The direction vector $\hat{n}$ represents the unit normal vector to the plane. Note that the accent ($\widehat{\ }$) represents that the vector is scaled to be a unit vector.) Finally, D represents the distance of the plane from the origin. Note that typically, if we don't scale the normal vector, then the vector equation of the plane is simply: $r \cdot n = D$, and in this situation the distance of the plane from the origin is $|\frac{D}{|n|}|$.

Here are examples of the same plane expressed in a Cartesian equation and a vector equation:

$$3x - 2y + 6z = 21$$
$$r \cdot (3i - 2j + 6k) = 21$$

Notice that if we scale the normal vector to be a unit vector, it would be $\frac{3}{7}i - \frac{2}{7}j + \frac{6}{7}k$. Thus, we could also write this equation as:

$$r \cdot \left(\frac{3}{7}i - \frac{2}{7}j + \frac{6}{7}k\right) = 3$$

Thus, this plane has distance 3 from the origin.

<u>Determining a cross product of three dimensional vectors</u>

Let the two vectors be ai+bj+ck and di+ej+fk. This computation can be done via the following determinant:

$$\begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ec)i + (cd - fa)j + (ae - db)k$$

In general, the 3 x 3 determinant above is equal to $ibf + jcd + kae - dbk - eci - faj$, using the "basket-weaving method" described here: http://home.cc.umanitoba.ca/~thomas/Courses/Determinants.pdf (page 71).

Example

Find the cross product (3i – 2j + 6k) x (i + j + 3k):

$$\begin{vmatrix} i & j & k \\ 3 & -2 & 6 \\ 1 & 1 & 3 \end{vmatrix} = (-6-6)i + (6-9)j + \big(3-(-2)\big)k = -12i - 3j + 5k$$

Note that the order is important. Specifically, if we swap the order of the vectors in the cross product, the result will be negated. Thus, (i+j+3k) x (3i – 2j + 6k) = 12i + 3j – 5k.

Determining the equation of a plane from three non-collinear points on the plane

If we are given three non-collinear points on the plane, we can create two non-parallel vectors on the plane. Then, by definition, if we take the cross product of these two vectors on the plane, we'll obtain the normal vector to both of them, which must ALSO be the normal vector to the plane. From here, there is only one unknown, namely the constant on the right hand side of the plane equation. In either form, we can solve for this constant by plugging in any of the points in the equation.

Example

What is an equation of a plane that goes through the points (2, 4, 6), (8, 1, 4) and (-2, 1, 6)?

Two vectors on the plane are 6i – 3j – 2k (movement from first point to second point) and -4i – 3j + 0k (movement from first point to third point).

Their cross product is $\begin{vmatrix} i & j & k \\ 6 & -3 & -2 \\ -4 & -3 & 0 \end{vmatrix} = -6i + 8j + (-18-12)k = -6i + 8j - 30k$

A multiple of this vector is 3i – 4j + 15k. Thus, an equation for this plane takes the form 3x – 4y + 15z = D. To find D, plug in a point on the plane. Let's choose (2, 4, 6);

$$3(2) - 4(4) + 15(6) = 80$$

Thus, an equation of the plane is $3x - 4y + 15z = 80$. Note that when we plug in the other two points into this equation, they satisfy the equation, showing that this equation is consistent with the points given.

Incidentally, the distance of this plane from the origin is $\frac{80}{\sqrt{3^2+4^2+15^2}} = \frac{16\sqrt{10}}{10}$.

Determining the intersection of three planes in the typical case

The typical intersection of three planes is a point. Atypical cases include no intersection because either two of the planes are parallel or all pairs of planes meet in non-coincident parallel lines, two or three of the planes are coincident, or all three planes intersect in the same line.

If we are given the information about the three planes as Cartesian equations, the point of intersection is simply the solution to the system of three equations in three variables. This can be solved via Cramer's rule:

$ax + by + cz = j$
$dx + ey + fz = k$
$gx + hy + iz = l$

Then $x = \dfrac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $y = \dfrac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$, $z = \dfrac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$

These solutions only hold, of course, if the denominator in each expression is non-zero. If this denominator is zero, then we're in one of the atypical cases described previously.

<u>Example</u>
Find the point at which the planes $2x + 4y + 3z = -14$, $x - 2y + 4z = -9$ and $-3x + y - 3z = 1$ intersect.

Using Cramer's rule, we find:

$$x = \frac{\begin{vmatrix} -14 & 4 & 3 \\ -9 & -2 & 4 \\ 1 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 4 & 3 \\ 1 & -2 & 4 \\ -3 & 1 & -3 \end{vmatrix}} = \frac{-84+16-27+6+56-108}{12-48+3-18-8+12} = \frac{-141}{-47} = 3$$

$$y = \frac{\begin{vmatrix} 2 & -14 & 3 \\ 1 & -9 & 4 \\ -3 & 1 & -3 \end{vmatrix}}{-47} = \frac{54+168+3-81-8-42}{-47} = \frac{94}{-47} = -2$$

$$z = \frac{\begin{vmatrix} 2 & 4 & -14 \\ 1 & -2 & -9 \\ -3 & 1 & 1 \end{vmatrix}}{-47} = \frac{-4+108-14+84+18-4}{-47} = \frac{188}{-47} = -4$$

Thus, the intersection of the three planes is (3, -2, -4).

Equations of Lines in Three Dimensions

Though the Cartesian equation of a line in three dimensions doesn't obviously extend from the two dimensional version, the vector equation of a line does. Thus, given a point, p_0, on the line and the line's direction vector, v, the equation of the line can be written as $r = p_0 + \lambda v$, just like in two dimensions.

Thus, a line through the point (5, -2, 4) with the direction vector i+ 2j + 3k has the vector equation:

$$r = (5, -2, 4) + \lambda(1, 2, 3)$$

As a parametric equation this is simply written as follows:

$$x = 5 + \lambda$$
$$y = -2 + 2\lambda$$
$$z = 4 + 3\lambda$$

If you wanted a Cartesian equation (which almost never gets used in programming representation of three dimensional lines), you can solve for λ in each of the equations above and set all three of these expressions equal: $x - 5 = \frac{y+2}{2} = \frac{z-4}{3}$. The denominators are the direction vector of the line and a point on the line is embedded in the numerator as the constants subtracted from x, y and z, respectively.

Intersection of a Plane and a Line

Now that we've defined equations of lines and planes in three dimensions, we can solve the intersection of the two. We solve the typical case as follows:

1) Get a parametric equation of the line
2) Substitute the right-hand sides of x, y and z into the plane equation.
3) Solve for λ, if possible. (If it's not possible, we're in a degenerate case.)
4) Plug in for λ in the line equation to determine the point of intersection.

The degenerate case corresponds to a situation where the line is parallel to the plane. Either, the line will be ON the plane, or it will not intersect with it at all. To determine which of the two scenarios is correct, pick any point on the line to plug into the plane equation. If it satisfies it, then the line is on the plane. If it doesn't, the line is parallel to the plane, but on a different plane.

Example

What is the intersection of the line (i + 4j + 9k) + (2i – j -4k)λ and the plane 3x – 4y + 6z = 13?

Rewrite the equation of the line parametrically: x = 1 + 2λ, y = 4 – λ, z = 9 – 4λ. Plug this into the plane equation:

$$3(1 + 2\lambda) - 4(4 - \lambda) + 6(9 - 4\lambda) = 13$$
$$3 + 6\lambda - 16 + 4\lambda + 54 - 24\lambda = 13$$
$$-14\lambda + 41 = 13$$
$$-14\lambda = -28$$
$$\lambda = 2$$

Thus, the point of intersection is x = 1 + 2(2) = 5, y = 4 – 2 = 2, z = 9 – 4(2) = 1, or (5, 2, 1).

Plane, Plane intersection

Typically, this is a line. It's not is when the normal vectors for both planes are parallel to each other. Otherwise, plug in an arbitrary value of x into both planes. This leaves you two equations in y and z. Solve for both of these for the point. Now, do this for a different value of x to yield a second point of intersection. Now, you have your line. The only time this won't work is when the intersection line is contained with a plane of the form x=c. You'll recognize this since the system of equations won't have a unique solution, meaning that a determinant in Kramer's rule will be zero. If this is the case, simply plug in two arbitrary values for y and solve for x and z. Example: Planes $x+3y+z = 12$ and $4x-5y+z = -8$. Plug in $x=0$ to yield $3y+z=12$ and $-5y+z =-8$. The solution here is $y=2.5$, $z=4.5$, soothe point (0, 2.5, 4.5) is on the line. Now, plug in $x=-2$ to yield $-2+3y+z = 12$ and $-8-5y+z = -8$. These simplify to $3y+z=14$ and $-5y+z = 0$. Solving, we get $y=1.75,z=8.75$, so the point (-2,1.75,8.75) is also on the line of intersection. The equation of this line of intersection is $r = (0, 2.5, 4.5) + t(-2, -.75, 4.25)$.

Distance between two parallel planes

In the equation $Ax + By + Cz = D$, the distance of the plane from the origin is $\frac{D}{\sqrt{A^2+B^2+C^2}}$. Recall that Ai + Bj + Ck is nothing but the normal vector to the plane for this equation. This means that if we scale our normal vector to be a unit normal vector, then the absolute value of D represents the distance of the plane from the origin. Thus, if we have two planes, $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$, then the distance between those planes is $\frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}$.

Distance between a point and a plane

Given a point (x_1, y_1, z_1) and a plane $Ax + By + Cz = D$, the plane parallel to the given plane through the point is: $Ax + By + Cz = Ax_1 + By_2 + Cz_2$. From here, just fine the distance between the two planes!

Distance between a line and a plane

If you attempt to intersect a line and a plane and don't get a unique solution, this means the line lies on a plane parallel to the given plane. We can find the distance between the two by taking any point on the line, determine the plane through that point parallel to the given plane and then finding the distance between the planes.

Note on calculating intersections of parametric equations

Sometimes the parameter in two parametric equations is the same, other times it is different. Make sure you know the difference. If we describe the motion of two different airplanes in terms of t, and are asked to calculate whether the airplanes crash, the parameter we plug into the two equations can't be different. (In real life, my car goes through the exact same spot in space as yours all the time, but hopefully never at the same exact time. Only the latter infers a wreck.) With the intersection of two lines, the two parameters are different.