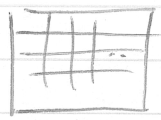
Arup's 2018 AMC 10B Solutions

1) A



$20/2 = 10$ columns
$18/2 = 9$ rows
$\overline{\quad 90 \text{ pieces}}$

2) D

1^{st} $\frac{1}{2}$ hr drove $60 \text{ mph} \times \frac{1}{2} \text{ hr} = 30$ miles
2^{nd} $\frac{1}{2}$ hr drove $65 \text{ mph} \times \frac{1}{2} \text{ hr} = 32.5$ miles
In 3^{rd} $\frac{1}{2}$ hr drove $96 - 30 - 32.5 = 33.5$ miles
AVG SPEED last $\frac{1}{2}$ hr $= \frac{33.5 \text{ m}}{.5 \text{ hr}} = 67$ mph

3) B

Choose $(1,2)$, $(1,3)$ or $(1,4)$ for 1^{st} pair
results in values 14, 11, 10, respectively
Only possible due to Comm Prop of $+$, $\times$.

4) B

$XY = 24$, $XZ = 48$, $YZ = 72$ mult together
$(XY)(XZ)(YZ) = 2^3 \times 3 \times 2^4 \times 3 \times 2^3 \times 3^2$
$X^2Y^2Z^2 = 2^{10} \times 3^4$, so $XYZ = 2^5 \times 3^2$
Thus $Z = \frac{XYZ}{XY} = \frac{32 \times 9}{8 \times 3} = 12$, $Y = 6$, $X = 4$, (same method)
$X + Y + Z = 4 + 6 + 12 = 22$

5) D

Primes $= \{2, 3, 5, 7\}$
subset with no primes $= 2^4$ (subs of $\{4, 6, 8, 9\}$)
Total subsets $= 2^8$, Ans $= 2^8 - 2^4 = 256 - 16 = 240$

6) D

Sample space of 1^{st} 2 draws is $5 \times 4 = 20$
Successes are $(1,2), (1,3), (2,1), (3,1)$; will all require
a third draw. (Rest don't); Prob $= \frac{4}{20} = \frac{1}{5}$.

7) D

Let small radius be r. Large radius is Nr.
Area small semicircles $= (\frac{1}{2}\pi r^2)N$.
Area large semicircle $= \cancel{\tfrac{1}{2}\pi}(Nr)^2 = 19(\cancel{\tfrac{1}{2}\pi} r^2)N$
$N^2 = 19N$
$N = 19$, since $N \neq 0$

8) C

1 level = 4 $\}+6$

2 levels = 10 $\}+8$

3 levels = 18

$$k \text{ levels} = 4+6+8+10+\dots(2k+2)$$
$$= (2+4+6+\dots(2k+2)) - 2$$
$$= 2[1+2+3+\dots(k+1)] - 2$$
$$= \frac{\cancel{2}(k+1)(k+2)}{\cancel{2}} - 2$$
$$= k^2+3k$$

Let $k^2+3k=180$

$$k^2+3k-180=0$$
$$(k-12)(k+15)=0$$
$$\boxed{k=12} \quad \cancel{k=-15}$$

9) D

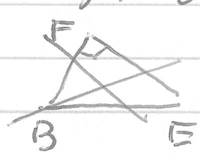
Min sum $= 7 + 3 = 10$

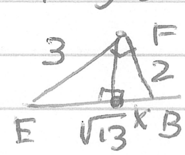
Max sum $= 42 - 3 = \boxed{39}$

Dice rolls like this are symmetric.
$1 \leftrightarrow 6$ $3 \leftrightarrow 4$
$2 \leftrightarrow 5$

10) E

Height of pyramid BCHEM is same as height of triangle FEB to base EB.





(hypotenuse of $\triangle$EB)

Using similar triangles $\triangle EFB$ to $\triangle FXB$

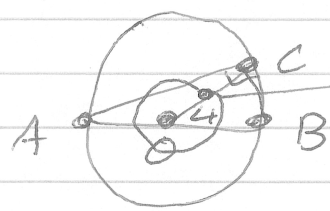
$$\frac{EF}{BE} = \frac{3}{\sqrt{13}} = \frac{FX}{2} \longrightarrow FX = \frac{6}{\sqrt{13}}$$

Area base BCHE $= \sqrt{13} \times 1 = \sqrt{13}$

Area pyramid(BCHEM) $= \frac{1}{3}\sqrt{13} \cdot \frac{6}{\sqrt{13}} = 2$

11) C Except for 5, primes end in 1, 3, 7, or 9 and their squares end in 1, 9, 9 or 1 (so just 1 or 9). More importantly all primes greater than 3 are 1 mod 3 or 2 mod 3. Their squares are always 1 mod 3. Add any number 2 mod 3 to p^2, and you'll get an integer divisible by 3. Thus, $p^2+26 \equiv 0 \pmod 3$ and isn't prime even. (Exception is $p=3$ but we can check that $3^2+26=35=5\times 7$.)

12) C



→ Centroid is $\frac{1}{3}$ of the way from O to C. Since $OC=12$, centroid pt is always 4 from O.

Locus is circle (minus 2 pts on AB) w/ radius 4. Area $= \pi 4^2 = 16\pi \sim 16 \times 3.14 \sim 50$

13) C

$1 \equiv 1 \bmod 101$

$10 \equiv 10 \bmod 101$

$100 \equiv -1 \bmod 101 \rightarrow 101 \equiv 0 \bmod 101$

$1000 \equiv -10 \bmod 101$

$10^4 \equiv 1 \bmod 101$

$10^5 \equiv 10 \bmod 101$

$10^6 \equiv -1 \bmod 101 \rightarrow 10^6+1 \equiv 0 \bmod 101$

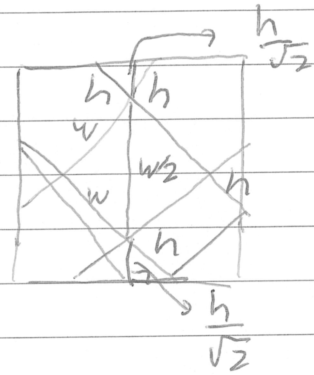
→ It follows that $10^2+1, 10^6+1, 10^{10}+1, \ldots, 10^{2018}+1$ are all divisible by 101. Add 2 to get 4, 8, ..., 2020 so 1, 2, ..., 505 terms

14) D

$$\begin{array}{r} 2018 \\ -10 \\ \hline 2008 \end{array} \qquad \left\lceil \frac{2008}{9} \right\rceil = 223+1 = 224$$

So 225 unique values minimum, since $10 + 223 \times 9 < 2018$ but $10 + 224 \times 9 \geq 2018$ ✓

15) A



$$S = \frac{h}{\sqrt{2}} + w\sqrt{2} + \frac{h}{\sqrt{2}}$$

$$S = \frac{2h}{\sqrt{2}} + w\sqrt{2} = \sqrt{2}(h+w)$$

$$S^2 = 2(h+w)^2$$

16) E If $a \equiv 0 \mod 2$, $a^3 \equiv 0 \mod 2$

If $a \equiv 1 \mod 2$, $a^3 \equiv 1 \mod 2$

If $a \equiv 0 \mod 3$, $a^3 \equiv 0 \mod 3$

If $a \equiv 1 \mod 3$, $a^3 \equiv 1 \mod 3$

If $a \equiv 2 \mod 3$ $a^3 \equiv 2^3 \equiv 2 \mod 3$

Thus, $a^3 \equiv a \mod 6$. (all combos 2,3 match)

Thus $\sum_{i=1}^{2018} a_i^3 \equiv \sum_{i=1}^{2018} a_i \equiv 2018^{2018} \mod 6$

$2018 \equiv 2 \mod 6$, thus

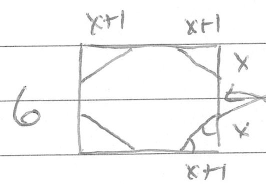
$2018^2 \equiv 4 \mod 6$, and

$2018^3 \equiv 2 \mod 6$

it follows that $2018^{2018} \equiv 4 \mod 6$

(alternating 2,4 odd, even powers)

17) B



45° since equilateral octagon angles are 135°

SIDE $= \sqrt{x^2 + (x+1)^2}$

$$x + \sqrt{x^2 + x^2 + 2x + 1} + x = 6$$

$$\left(\sqrt{2x^2+2x+1}\right)^2 = (6-2x)^2$$

$$2x^2 + 2x + 1 = 4x^2 - 24x + 36$$

$$2x^2 - 26x + 35 = 0$$

$$x = \frac{26 \pm \sqrt{676 - 4 \cdot 2 \cdot 35}}{2 \cdot 2} = \frac{13}{2} - \frac{\sqrt{396}}{4} = \frac{13 - \sqrt{99}}{2}$$

$$S = 6 - 2x = 6 - 2\left(\frac{13 - \sqrt{99}}{2}\right) = 6 - (13 - \sqrt{99}) =$$

$$-7 + \sqrt{99} = \boxed{-7 + 3\sqrt{11}}$$

$$\boxed{\begin{array}{r} -7 \\ +3 \\ +11 \\ \hline 7 \end{array}}$$

18) D Label as $A_1 A_2 B_1 B_2 C_1 C_2$.
1^{st} count 2 pairs of sibs in same row.

$A_1\ C_1\ A_2$
$C_2\ B_1\ B_2$ oops can't happen!

Must have $A_1\ B_1\ C_1$ / $B_2\ C_2\ A_2$ — Perm of one set, Derangement for other set.

Pick 1^{st} row is $2^3 \times 3! = 48$ ways
So 2 choices for each pair, then permute.
Second row can be arranged in 2 ways, the 2 derangements. (If 1^{st} row is $B_2\ C_1\ A_1$, then 2^{nd} row must be A_2, B_1, C_2 OR $C_2\ A_2\ B_1$.)
Final answer is $48 \times 2 = 96$

19) E Let d be difference in Chloe + Zoe's ages
We must have 9 integers x for which $\underset{\text{Zoe}}{x} \mid \underset{\text{Chloe}}{(x+d)}$. This occurs if and only if $x \mid d$. Thus d must have 9 divisors. $d = p_1^{a_1} p_2^{a_2}$ with $(a_1+1)(a_2+1) = 9$, (OR $d = p_1^{a_1}$ and $a_1 = 8$ which isn't really possible.)
So $a_1 = 2, a_2 = 2$ and to be a reasonable new parent, we have $p_1 = 2, p_2 = 3$ and $d = 2^2 \times 3^2 = 36$. Thus, currently Chloe $= 37$, Joey $= 38$. Since Chloe + Joey have a difference in age of 37 and $37 \in$ Prime, Only when Chloe $= 37$, Joey $= 74$ will Joey's age be a multiple of Chloe's.
$7 + 4 = 11$.

20) B List out the 1st few terms:

1, 1, 3, 6, 8, 8.

Notice that $f(5)=f(6)$. Let's investigate what happens if $f(n)=n+3$ and $f(n+1)=n+3$:

$f(n+2) = f(n+1) - f(n) + n+2 = n+2$

$f(n+3) = (n+2) - (n+3) + n+3 = n+2$

$f(n+4) = (n+2) - (n+2) + n+4 = n+4$

$f(n+5) = (n+4) - (n+2) + n+5 = n+7$

$f(n+6) = (n+7) - (n+4) + n+6 = \boxed{n+9}$

$f(n+7) = (n+9) - (n+7) + n+7 = \boxed{n+9}$

This cycle of 2 consecutive equal terms repeats every 6 terms.

More specifically, if $n \equiv 5 \mod 6$, then $f(n) = f(n+1) = n+3$.

It follows that $f(2015) = f(2016) = 2018$

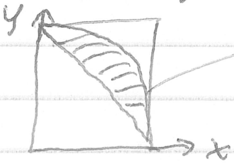
$f(2017) = 2017$ and $f(2018) = 2017 - 2018 + 2018$

$= \boxed{2017}$

21) C, Mary's number n is a multiple of $2 \times 323 = 2 \times 19 \times 17$. Since n has 4 digits $n = 2 \times 17 \times 19 \times m$, where $2 \leq m \leq 15$. (If $m=1$, n has 3 digits, if $m=16$, n has 5 digits.)

To create a divisor > 323, we could try $17 \times (2m)$ or $17 \times m$ or $19 \times m$ or $19 \times (2m)$. Notice that automatically $17m$, $19m$ are too small. Of the other two forms since $17 < 19$, we can do $17 \times 2 \times 10 = 340$, this is best because the first multiple of 19 greater than 323 is 342.

22) C, Display the sample space as a unit square:



$\to x+y=1$.

if $x+y<1$, no triangle is formed!

if $x^2+y^2<1$ and $x+y\geq 1$, then a triangle is formed but it's obtuse, since $x^2+y^2=1$ would make a right triangle.

The desired probability is the shaded area above where $x^2+y^2<1$ and $x+y\geq 1$. This is simply $\frac{1}{4}\times\pi\times 1^2-\frac{1}{2}\times 1\times 1=\frac{\pi}{4}-\frac{1}{2}\approx .79-.5$

$4\,\overline{)3.14159}$ $.7854$

$\approx .29$ to nearest hundredth.

23) B Let $L=\text{lcm}(a,b)$ and $G=\gcd(a,b)$. Recall that $a\times b=\text{lcm}(a,b)\times\gcd(a,b)$. (Usually people learn it the other way around!) Now substitute:

$$ab=LG\overset{+63}{=}20L+126$$
$$LG-20L-126=-63$$
$$LG-20L-126+240=240-63$$
$$(L-12)(G-20)=177=3\times 59$$

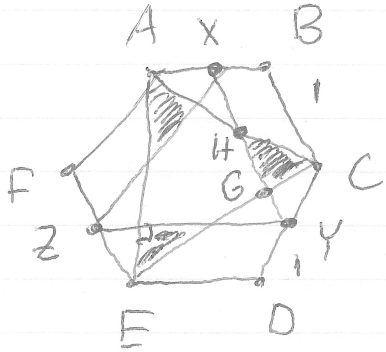
Try:

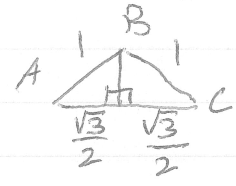
$L-12=59$	$L-12=177$
$G-20=3$	$G-20=1$
$G=23, L=71$	$G=21, L=189$
No DIV X	$21\times 9=189$

try $\gcd(a,b)=21$
$\text{lcm}(a,b)=189$

Definitely $a=21, b=189$ and $a=189, b=21$ WORK. Since $\frac{189}{21}=9$ and $9=3\times 3$ only but $\gcd(3,3)\neq 1$. ~~Thus~~ ~~is~~ these are the only 2 solutions.

24) C





$AC = \sqrt{3}$ Since ~~#~~ $\triangle ABC$ is two $30°-60°-90°$ Δs

$$Area(ACE) = \frac{1}{4}(\sqrt{3})^2\sqrt{3} = \frac{3\sqrt{3}}{4}.$$

Note that all 3 shaded triangles are congruent. We need to find the area of GHC and multiply it by 3, the subtract this out from Area(ACE).

~~If ED is parallel to the x-axis~~

BC ∥ XY due to placement. Since AC is a transversal to BC and XY, $\measuredangle AHX = \measuredangle CHY = \measuredangle BCA = 30°$, also $\measuredangle ACE = 60°$ so $\measuredangle HGC = 90°$. It follows that both ΔHGC and ΔCGY are $30°-60°-90°$ triangles.

$$CY = \frac{1}{2} \to GY = \frac{1}{4} \to GC = \frac{\sqrt{3}}{4}, \text{ then } GC = \frac{\sqrt{3}}{4} \to GH = \frac{\sqrt{3}}{4} \times \sqrt{3} = \frac{3}{4}$$

$$Area(HGC) = \frac{1}{2} \times \frac{3}{4} \times \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{32}$$

$$Area(\text{Hexagon of Intersection}) = \frac{3\sqrt{3}}{4} - 3\left(\frac{3\sqrt{3}}{32}\right) = \frac{24\sqrt{3}}{32} - \frac{9\sqrt{3}}{32} = \boxed{\frac{15\sqrt{3}}{32}}$$

251C, Let $x = \lfloor x \rfloor + \Delta$, where $0 \le \Delta < 1$

$$(\lfloor x \rfloor + \Delta)^2 + 10000 \lfloor x \rfloor = 10000 (\lfloor x \rfloor + \Delta)$$
$$x^2 = 10000\Delta$$

Consider x as x ranges from n to $n+1$, where $-99 \le n \le 99$. The left-hand side "moves" from n^2 to $(n+1)^2$ and the right-hand side moves from 0 to $10000 - \epsilon$. $\epsilon > 0$ but is infinitely small. It's guaranteed that the RHS < LHS at first but eventually ~~the~~ RHS > LHS meaning there is an intersection point. For $x=0$, $\Delta=0$ matches. Thus, there are $99-(-99)+1 = 199$ solutions