# <u>OMC Number Theory Solutions (for problems from 10/14/2025)</u>

1) **Problem (2006 AMC 10B #22)**

N(4B + 5J) = 253
N(4B+5J) = 11 x 23

Since N > 1, N = 11 and 4B + 5J = 23. The only integer solution to the latter is 4(2) + 5(3) = 23, so B = 2 and J = 3. (Note: N can't be 23 because there are no positive integer solutions to 4B + 5J = 11.)

Thus the total cost of jam for all 11 sandwiches is 11 x 5 x 3 = 165 cents. **(D)**

2) **Problem (2006 AMC 10B #25)**

Since all kids have different ages and the oldest kid is 9, assuming that no kid is 0, then the kids must have every different age except for one in between 1 and 9. So, we must form some four digit number which contains each of two different digits twice, such that it's divisible by 7 integers from the list: 2, 3, 4, 5, 6, 7, 8 and 9, with one of those integers being 9. We can assume that the number is divisible by 2 and 4 since if it wasn't divisible by 4, it wouldn't be divisible by 8. So at a minimum, the number is divisible by 4 x 9 = 36. (This covers the ages 2, 3, 4, 6, and 9.) Thus, our answer has to be 5, 7 or 8. Our goal would be to choose 2 out of these three numbers such that the lcm of 36 and the two chosen numbers divides evenly into the aforementioned 4 digit number.

If we choose 5, this is a rather tricky situation because 36 x 5 = 180, meaning that all multiples end in 0 so that our end answer would have to have two digit 0s and either be of the four x0x0 or xx00. Numbers of the form x0x0 are divisible by 101 (can you see why?) This would mean that the number is either 9090 or 9900, the only numbers of either form divisible by 9. Neither of these is divisible by 7 or 8, so we can confidently say that 5 is the number that isn't one of the kids' ages.

Note: We can verify a possibility by finding a multiple of 1, 2, 3, 4, 6, 7, 8 and 9, which means a multiple of 36 x 2 x 7 = 504 which satisfies the criteria. Namely 504 x 11 = 5544. It follows that Mr. Jones is 44. **(B)**

3) **Problem (2007 AMC10A #17)**

$75m = n^3$
$3 \times 5^2 \times m = n^3$, at a minimum both 3 | n and 5 | n. Try = 15. (Minimizing n also minimizes m.)
$3 \times 5^2 \times m = 3^3 \times 5^3$
$m = 9 \times 5 = 45$

m + n = 45 + 15 = 60. **(D)**

4) **Problem (2007 AMC10A #23)**

$m^2 - n^2 = 96$
$(m + n)(m - n) = 2^5 \times 3$ (so this has 6 x 2 = 12 divisors)

The smallest divisors are: 1, 2, 3, 4, 6 and 8

Try m – n = 1 --> m + n = 96, no solution
m – n = 2 --> m + n = 48, 1 solution
m – n = 3 --> m + n = 32, no solution
m – n = 4 --> m + n = 24, 1 solution
m – n = 6 --> m + n = 16, 1 solution
m – n = 8 --> m + n = 12, 1 solution

Total of four solutions with $m \geq n$. **(B)**

5) **Problem (2007 AMC 12A #11)**

Each digit is guaranteed to appear in each spot the same number of times. So the sum of all of these numbers is divisible by 111. This is because we can rewrite the sum as the sum of numbers from the set {111, 222, 333, 444, 555, 666, 777, 888, 999}. Since once a digit appears somewhere in the sequence, it must appear in the other two locations as well. 111 = 3 x 37, so the answer to the question is 37. It's easy enough to produce a sequence with this largest prime divisor. Just do 123, 231, 312. **(D)**

6) **Problem (2007 AMC 12A #12)**

For the difference to be even, there are two possibilities, both terms are even or both terms are odd. The probability one term is odd is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, since both variables in the product must be odd and each has a probability of a half of being odd. Thus, the probability that both terms are odd is $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$. (This corresponds to when all four of a, b, c and d are odd.) Alternatively, the probability of a term being even is $1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$. The probability that both terms are even is just $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$.

It follows that the desired result is $\frac{1}{16} + \frac{9}{16} = \frac{10}{16} = \frac{5}{8}$. **(E)**

7) **Problem (2007 AMC 12A #22)**

n + S(n) + S(S(n)) = 2007

First note that S(n) is smaller than n for all n > 9. Thus, the dominant part of the sum has to be the first term. If we assume n is 4 digits, then $S(n) \leq 36$ and $S(S(n)) \leq 11$, (if S(n) = 29...) This proves that $n \geq 2007 - 36 - 11 = 1960$.

Let's just case out the first digit of n. Try 200x. The only solution here is n = 2001. (n = 2000 results in the LHS being too small, 2001 works, and anything above 2002 creates a LHS with a sum greater than 2007.

Otherwise, n = 19xy. S(n) = 1 + 9 + x + y = 10 + x + y and S(S(n)) = 1 + x + y, if x+y < 10, or S(S(n)) = 2 + x + y – 10 = x + y - 8, if x + y ≥ 10.

Adding 19xy + 10 + x + y = 1900 + 10x + y + 10 + x + y = 1910 + 11x + 2y

If x+y < 10, then n + S(n) + S(S(n)) = 1910 + 11x + 2y + 1 + x + y = 1911 + 12x + 3y

1911 + 12x + 3y = 2007
12x + 3y = 96, has the solution x = 8, y = 0

If x + y≥ 10, then n + S(n) + S(S(n)) = 1910 + 11x + 2y + x + y – 8= 1902 + 12x + 3y

1902 + 12x + 3y = 2007
12x + 3y = 105, the solutions are x = 8, y = 3, OR x = 7, y = 7

Thus, there are 4 possible values of n. n = 2001, n = 1980, n = 1983 and n = 1977

2001 + 3 + 3 = 2007
1980 + 18 + 9 = 2007
1983 + 21 + 3 = 2007
1977 + 24 + 6 = 2007
**(D)**

8) **Problem (1992 AHSME #17)**

We need to check for divisibility by 3, 9 and 27, in that order, based on the answer choices.

Let's sum the digits in the integers from 19 through 92, inclusive.

19 digit sum is 10
20 to 29 has a digit sum of 2(10) + 45
30 to 39 has a digit sum of 3(10) + 45
40 to 49 has a digit sum of 4(10) + 45
50 to 59 has a digit sum of 5(10) + 45
60 to 69 has a digit sum of 6(10) + 45
70 to 79 has a digit sum of 7(10) + 45
80 to 89 has a digit sum of 8(10) + 45
90 to 92 has a digit sum of 30

Adding we get 10 + (2 + 3 + 4 + 5 + 6 + 7 + 8)(10) + 7(45) = 10 + 350 + 315 = 375.

The digit sum here is divisible by 3 but not 9, which means the original number is divisible by 3 but not 9. **(B)**

9) **Problem (2000 AIME #1)**

$10^n = 2^n$ x $5^n$, any other product split would automatically have a trailing zero (product of 2 an 5). Thus, we need to work out cases for both $2^n$ and $5^n$ via brute force.

Powers of 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, so answer is 10 or less.

Powers of 5: 5, 25, 125, 625, 3125, 15625, 78125, 390625, so answer is **8.**

10) **Problem (2003 AMC 10A #25)**

Let n = abcde, where a, b, c, d and e are the respective digits.

q = abc
r = de
q + r = 100a + 10b + c + 10d + e

Take this mod 11 to get:

$100a + 10b + c + 10d + e \equiv a - b + c - d + e \pmod{11}$

If this is equivalent to 0 (mod 11) then q + r is divisible by 11. But, this is precisely the divisibility criteria for n = abcde to be divisible by 11. Thus, the answer to the question is simply the number of 5 digit integer divisible by 11. Note that $\left\lceil \frac{10000}{11} \right\rceil = 910$ and $\left\lfloor \frac{99999}{11} \right\rfloor = 9090$.

It follows that the number of 5-digit integers divisible by 11 is 9090 – 910 + 1 = 8181. **(B)**