OMC Lecture Problems 10/14/2025 – Number Theory Solutions

1. Problem (2018 AMC 10A #17)

Let S be a set of 6 integers taken from $\{1, 2, \dots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible value of an element in S?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 7$

1 is easily not possible (nothing else could get added)

2 --> 4,6,8,10,12 banned, so we would need 3,5,7,9,11 all (but 9 is mult of 3)

3 --> 6,9,12 banned, [4,5,7,8,10,11], can only have 1 yellow and 1 blue, makes our set size too small

4 --> 4, 5, 6, 7, 9, 11 (okay) --> **choice C**

2. Problem (2018 AMC 10A #19)

A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \dots, 2018\}$. What is the probability that m^n has a units digit of 1?

$\textbf{(A)}\ \frac{1}{5} \qquad \textbf{(B)}\ \frac{1}{4} \qquad \textbf{(C)}\ \frac{3}{10} \qquad \textbf{(D)}\ \frac{7}{20} \qquad \textbf{(E)}\ \frac{2}{5}$

1 - 1, 1, 1, 1, ... 20 out 20
3 - 3, 9, 7, 1, 5 out 20
5 - 5, 5, 5, 5.. 0 out 20
7 - 7, 9, 3, 1 5 out 20
9 - 9, 1, 9, 1 10 out of 20
All cycles repeat in 4 steps, furthermore 1999 to 2018, has 20 consecutive integers. Doesn't matter where we start in the cycle, the possible exponents do a complete number of cycles so we don't have to look at the end points.

40 out of 100 --> 2/5 **(E)**

3. Problem (2018 AMC 12B #15)

How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

$\textbf{(A)}\ 96 \qquad \textbf{(B)}\ 97 \qquad \textbf{(C)}\ 98 \qquad \textbf{(D)}\ 102 \qquad \textbf{(E)}\ 120$

_ _ 1 (x + y = 2 mod 3, leading digit isn't 0, and no 3s)
_ _ 5 (x + y = 1 mod 3, leading digit isn't 0, and no 3s)
_ _ 7 (x + y = 2 mod 3, leading digit isn't 0, and no 3s)
_ _ 9 (x + y = 0 mod 3, leading digit isn't 0, and no 3s)

2, 5, 8, 11, 14, 17 (sum of digits)

2 = 20, 11
5 = 50, 41, 14
8 = 80, 71, 62, 44, 26, 17
11 = 92, 74, 65, 56, 47, 29
14 = 95, 86, 77, 68, 59
17 = 98, 89

24 options

1, 4, 7, 10, 13, 16,
1 = 10
4 = 40, 22
7 = 70, 61, 52, 25, 16
10 = 91, 82, 64, 55, 46, 28, 19
13 = 94, 85, 76, 67, 58, 49
16 = 97, 88, 79

24 options

3, 6, 9, 12, 15, 18

3 = 21, 12
6 = 60, 51, 42, 24, 15
9 = 90, 81, 72, 54, 45, 27, 18
12 = 84, 75, 66, 57, 48
15 = 96, 87, 78, 69
18 = 99

24 options

3 x 24 + 24 = 96 (A)

Another option for counting this is doing buckets for each digit. For example, for items of the form xy1, if x is 0 mod 3, then y is 2 mod 3, if x is 1 mod 3, y is 1 mod 3 and if x is 2 mod 3, y is 0 mod 3. For x there are 2 choices that are 0 mod 3, 3 choices that are 1 mod 3 and 3 choices that are 2 mod 3. For y there are 3 choices for each. Thus, this sum of products is

2 x 3 + 3 x 3 + 3 x 3 = 24

Yeah, this is probably faster!

4. Problem 24 (AMC 8 #24)

The digits 1, 2, 3, 4, and 5 are each used once to write a five-digit number $PQRST$. The three-digit number PQR is divisible by 4, the three-digit number QRS is divisible by 5, and the three-digit number RST is divisible by 3. What is P?

$\textbf{(A) } 1 \qquad \textbf{(B) } 2 \qquad \textbf{(C) } 3 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 5$

S has to be 5. Then R + T is either 1, 4 or 7. It can't be 1, so it's 4 or 7. 4 = 3 + 1 and 7 = 4 + 3, so either way 3 is either R or T, the other is 1 or 4. 2 must be P or Q. Since PQR is divisible by 4, R can not be 1 or 3, so R must be 4 which means that T is 3. So our number is PQ453. To make PQR divisible by 4, we must have Q = 2, so the number is 12453.

The answer choice is **A.**

5. Problem 16 (AMC 12B #16)

In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

$\textbf{(A) } 1 \qquad \textbf{(B) } 3 \qquad \textbf{(C) } 5 \qquad \textbf{(D) } 6 \qquad \textbf{(E) } 7$

Let's break into odd and even cases.

For odd, Let there be 2a+ 1 numbers that are x – a, x – a + 1, ..., x, x+1, ..., x+a. The sum of these numbers is (2a+1)x = 345 = 3 x 115 = 3 x 5 x 23, thus 2a + 1 = 1,3,5,15,23,69,115,345, but we must also have a > 0 and x – a > 0, the solutions are a = 1, 2,7,11

For even, we have 2a integers x – a + 1, x – a + 2, ..., x, x + 1, x + a = 2ax + a = a(2x+1). Here we need x – a + 1 > 0 and a > 0, so solutions are a = 1, 3, 5, when a = 15 2x + 1 = 23 and x = 11 so x – a + 1 > 0 isn't satisfied.

It follows that there are 7 possible solutions. Choice **E.**

6. Problem (2008 iTest #47)

Find $a+b+c$, where a, b, and c are the hundreds, tens, and units digits of the six-digit integer $123abc$, which is a multiple of 990.

c = 0, since the integer is divisible by 10.
Since it's divisible by 9, a + b + 0 = 3 or 12 (21 is too big)
Since it's divisible by 11, 0 – b + a – 3 + 2 – 1 is also divisible by 11
Thus -b+a = -9 or -b+a = 2. The former could only happen if b=9 and a=0 but then the first sum isn't satisfied. It follows that:
a + b = 12
a – b = 2
Thus, 2a = 14 and a = 7 and b = 5. The desired number is 123750 and the answer is
7 + 5 + 0 = **12**

7. Problem (2010 AIME I #2)

Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99\cdots9}_{\text{999 9's}}$ is divided by 1000.

All numbers that end in 999 are equivalent to -1 mod 1000. It follows that the product above, taken mod 1000 is

9 x 99 x (-1) x (-1) … x (-1), the number of (-1) terms is 997, so an odd number of them
= 9 x 99 x (-1) mod 1000
= -891 (mod 1000)
= 109 (mod 1000) (added 1000 which doesn't change the mod…)

Answer is **109.**

8. Problem (2005 AMC10B #24)

Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x+y+m$?

$\textbf{(A)}\ 88 \qquad \textbf{(B)}\ 112 \qquad \textbf{(C)}\ 116 \qquad \textbf{(D)}\ 144 \qquad \textbf{(E)}\ 154$

x = 10a + b, y = 10b + a
$(10a + b)^2 – (10b + a)^2 = m^2$
$100a^2 + 20ab + b^2 – 100b^2 – 20ab – a^2 = m^2$
$99a^2 – 99b^2 = m^2$
$9 \times 11 \times (a + b)(a – b) = m^2$

Either a+b or a-b is 11, otherwise we can't get 11^2 in the factorization of the left hand size. It can't be a bigger number since a and b are digits. It's not a-b, so set a+b = 11. That means that a-b is a perfect square, since 9 is already a perfect square. The only way a+b = 11 and a-b is a perfect square is if a-b = 1 (4 doesn't work because it leads to non-integer digits), thus, a = 6 and b = 5. It follows that x = 65. y = 56 and m = 3 x 11 = 33.

The corresponding sum is 56 + 65 + 33 = 154, choice **E.**

9. Problem (2005 AIME 2 #4)

Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.

$10^{10} = 2^{10} \times 5^{10}$
$15^7 = 3^7 \times 5^7$
$18^{11} = 2^{11} \times 3^{22}$, since $9 = 3^2$

We must use the inclusion-exclusion principle.
Let set A be the divisors of 10^{10}, set B be the divisors of 15^7 and set C be the divisors of 18^{11}.
Note: |A| means the size of set A

|A| = 11 x 11 = 121
|B| = 8 x 8 = 64
|C| = 12 x 23 = 276
|A intersect B| = divisors of 5^7, there are 8 of these
|A intersect C| = divisors of 2^{10}, there are 11 of these
|B intersect C| = divisors of 3^7, there are 8 of these
|A intersect B intersect C| = divisors of 1, there are 1 of these.

Our final count is

121 + 64 + 276 – 8 – 11 – 8 + 1 = **435**

Answers

1. C
2. E
3. A
4. A
5. E
6. 12
7. 109
8. E
9. 435