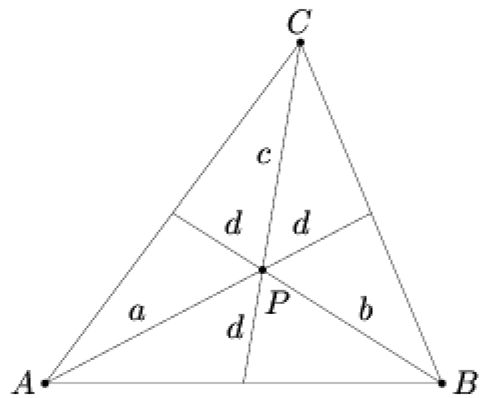
1988 Problem 12

Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a+b+c=43$ and $d=3$.



Solution (441)

Let CP meet AB and D. If we use AB as the base of the triangle, then the area of APB divided by the area of ACB is equal to $\frac{d}{c+d}=\frac{3}{c+3}$. This is because both triangles have the same base so their areas are proportional to their heights and the heights are proportion PD to CD. Similarly, we can set up the area of BCP divided by area BCA to be $\frac{3}{a+3}$. Finally, the area of ACP divide by the area of ACB is $\frac{3}{b+3}$. Let [ABC] denote the area of triangle ABC.

Notice that $\frac{[ABP]}{[ABC]}+\frac{[BCP]}{[BCA]}+\frac{[ACP]}{[ACB]}=\frac{[ABP]+\{BCP]+\{ACP]}{[ABC]}=\frac{[ABC]}{[ABC]}=1$.

It follows that:

$$\frac{3}{a+3}+\frac{3}{b+3}+\frac{3}{c+3}=1$$

Multiply this equation through by (a + 3)(b + 3)(c + 3):

$$3(a+3)(b+3)+3(a+3)(c+3)+3(b+3)(c+3)=(a+3)(b+3)(c+3)$$

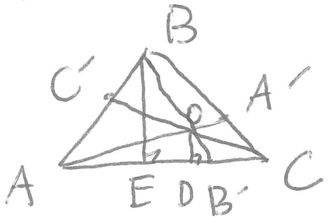
$$3(ab+3(a+b)+9+ac+3(a+c)+9+bc+3(b+c)+9)=(a+3)(b+3)(c+3)$$

$$3(ab+ac+bc)+18(a+b+c)+81=abc+3(ab+ac+bc)+9(a+b+c)+27$$

$$abc=54+9(a+b+c)=54+9\times 43=54+387=\mathbf{441}$$

1992 Problem 14 $\boxed{094}$

In triangle ABC, A', B', and C' are on the sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that $\frac{AO}{OA'}+\frac{BO}{OB'}+\frac{CO}{OC'}=92$, find $\frac{AO}{OA'}\cdot\frac{BO}{OB'}\cdot\frac{CO}{OC'}$.



Let $[ABC]$ denote area of $\triangle ABC$.

Let $x=[AOB]$, $y=[AOC]$, $z=[BOC]$.

$\triangle ODB' \sim \triangle BEB'$, thus $\frac{BE}{BB'}=\frac{OD}{OB'}$ and

$\frac{BB'}{OB'}=\frac{BE}{OD}$.

$[AOC]=\frac{1}{2}(AC)(OD)$ $\quad [ABC]=\frac{1}{2}(AC)(BE)$

$$\frac{x+y+z}{y}=\frac{[ABC]}{[AOC]}=\frac{\frac{1}{2}(AC)(BE)}{\frac{1}{2}(AC)(OD)}=\frac{BE}{OD}=\frac{BB'}{OB'},\ \frac{BB'}{OB}-\frac{OB'}{OB'}=\frac{BO}{OB'}=\frac{x+y+z}{y}-\frac{y}{y}=\frac{x+z}{y}$$

Thus, $\frac{BO}{OB'}=\frac{x+z}{y}$, Similarly $\frac{AO}{OA'}=\frac{x+y}{z}$ and $\frac{CO}{OC'}=\frac{y+z}{x}$.

$\frac{x+z}{y}+\frac{x+y}{z}+\frac{y+z}{x}=92$. We must find

$$\left(\frac{x+z}{y}\right)\left(\frac{x+y}{z}\right)\left(\frac{y+z}{x}\right)=\frac{(x+y)(x+z)(y+z)}{xyz}$$

$$=\frac{x^2y+x^2z+y^2x+y^2z+z^2x+z^2y+2xyz}{xyz}$$

$$=\frac{x^2(y+z)+y^2(x+z)+z^2(x+y)}{xyz}+2$$

Note that $\frac{x+z}{y}+\frac{x+y}{z}+\frac{y+z}{x}=\frac{(x+z)xz+xy(x+y)+yz(y+z)}{xyz}=\frac{x^2y+x^2z+y^2x+y^2z+z^2x+z^2y}{xyz}$

Thus, our final answer is

$= 92+2 = \boxed{094}$