Number Theory – AIME Preparation OMC Homework for 1/17/2025

Problems from Old AIMES

1992-15) Define a positive integer n to be a factorial tail if there is some positive integer m such that the base ten representation of m! ends with exactly n zeroes. How many integers less than 1992 are not factorial tails?

1994-5) Given a positive integer n, let p(n) be the product of the non-zero digits of n. (If n has only one digit, then p(n) is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999)$$

What is the largest prime factor of S?

1995-10) What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a composite number?

2005-12) For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(n) = 1$ and $\tau(6) = 4$. Define S(n) by

$$S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$$

Let a denote the number of positive integers $n \leq 2005$ with S(n) odd, and let b denote the number of positive integers $n \leq 2005$ with S(n) even. Find $|a - b|$.

Problems from recent AIMEs

2020-I-4) Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, 42,020 is in S because 4 is a divisor of 42,020. Find the sum of all the digits of all the numbers in S. For example, the number 42,020 contributes 4 + 2 + 0 + 2 + 0 = 8 to this total.

2022-I-7) Let a, b, c, d, e, f, g, h and i be distinct integers from 1 to 9. The minimum possible positive value of

$$\frac{a \times b \times c - d \times e \times f}{g \times h \times i}$$

can be written as m/n, where m and n are relatively prime positive integers. Find m + n.

2023 AIME-I 4) The sum of all positive integers m such that $\frac{13!}{m}$ is a perfect square can be written as $2^a3^b5^c7^d11^e13^f$, where a, b, c, d, e and f are positive integers. Find a + b + c + d + e + f.

2023 AIME-II 5) Let S be the set of all positive rational numbers r such that when the two numbers r and 55r are written as fractions in lowest terms, the sum of the numerator and denominator of one fraction is the same as the sum of the numerator and denominator of the other fraction. The sum of all the elements of S can be expressed in the form p/q, where p and q are relatively prime positive integers. Find p + q.