Number Theory – AIME Preparation OMC Homework for 1/17/2025 Solutions

Problems from Old AIMES

1992-15) Define a positive integer n to be a factorial tail if there is some positive integer m such that the base ten representation of m! ends with exactly n zeroes. How many integers less than 1992 are not factorial tails?

Solution (Ans = 396)

In general, the number of zeros at the end of n! equals the number of times the prime number 5 divides evenly into n! because the prime factorization of n! has more copies of the prime number 2 than 5. Looking at the definition of n! and "crossing out" copies of 5 as a factor, we find that the number of zeros at the end of n! is $\sum_{i=1}^{n} \left\lfloor \frac{n}{5^i} \right\rfloor$. We can calculate this for a given value of n via repeated division by 5. Let's make a table for the first few values of this sum, noticing that changes only occur at multiples of 5. Let f(n) equal the number of 0s at the end of n! Namely f(n) = $\sum_{i=1}^{n} \left\lfloor \frac{n}{5^i} \right\rfloor$.

n	5	10	15	20	25	30	35	40	45	50	55
f(n)	1	2	3	4	6	7	8	9	10	12	13

So, values get skipped when we transition between n and n+1, where n+1 is a multiple of 5^2. More generally, if the most number of times 5 divides evenly into n+1 is k, then k – 1 values get skipped as factorial tails.

Ultimately, we should find the first number on this chart greater than 1991. Notice that the given sum is less that or equal to the infinite sum without the floor function. The value of that sum would be $\frac{n/5}{1-1/5} = \frac{n}{4}$. Setting this to 1992, we find n = 7968. Let's use our algorithm to see how many zeroes are at the end of 7970:

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5 | 7970
5 | 1594     R 0
5 |  318     R 4
5 |   63     R 3
5 |   12     R 3
5 |    2     R 2
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zeros at end of 7970! = 1594 + 318 + 63 + 12 + 2 = 1989

This means that 7975! will have 1989 + 2 = 1991 zeros at the end of it because 7975 is divisible by 25 but not 125.

Now, we must figure out how many unique values f(n) equals for n in the range [5, 7979]. Then we can subtract this value from 1991 to get our final answer.

Notice that each range of n of the form [5k, 5k+4] has the same value for f(n) and that for each different integer k, the output is different. Thus, we can answer our question by simply answering how many ranges of the form [5k, 5k + 4] constitute the range [5, 7979]. The first range is for k = 1 while the last range is for k = 1595. Our final answer is 1991 – 1595 = **396.**

1994-5) Given a positive integer n, let p(n) be the product of the non-zero digits of n. (If n has only one digit, then p(n) is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999)$$

What is the largest prime factor of S?

Solution (Ans = 103)
Consider the product (1 + 2 + 3 + … + 9)(1 + 2 + 3… + 9)(1 + 2 + 3 + .. + 9). In this product, every term is the product of digits of the 729 integers that can be formed with digits 1 through 9 that are 3 digits long. This is almost exactly what we want, except that we aren't allowing for numbers with a digit of 0. But instead of multiplying by 0 we want to multiply by 1 for the multiplicative identity, since we want to ignore 0. So, adjust our sums within our product like so:

(1 + 1 + 2 + 3 + … + 9)(1 + 1 + 2 + 3… + 9)(1 + 1 + 2 + 3 + .. + 9).

This sum has a 1000 terms, of which each term is the desired product to be added except that this sum has the term 1 x 1 x 1 for the number 0. Since this isn't included in our sum, the value of S is actually

(1 + 1 + 2 + 3 + … + 9)(1 + 1 + 2 + 3… + 9)(1 + 1 + 2 + 3 + .. + 9) – 1 = $46^3 - 1$.

Let's factor this:

$(46 - 1)(46^2 + 46 + 1) = 3^2 \times 5 \times 2163 = 3^2 \times 5 \times 3 \times 721 = 3^3 \times 5 \times 7 \times 103$

The largest prime in this product is **103.**

1995-10) What is the largest positive integer that is not the sum of a positive integral multiple of 42 and a composite number?

Solution (Ans = 215)
Consider N = 42a + b. If 42 < N < 84, b = N – 42, and for several values of N, this is prime.
If 84 < N < 126, then b = N – 42 or N – 84. Ultimately, we have a list of several values and have to ask the question: is it possible for each of these values to be prime?

N – 42, N – 84, N – 126, N – 168, N – 210.

Let's consider each of these values mod 5. We select 5 because gcd(5, 42) = 1.

$42 \equiv 2 \pmod 5$
$84 \equiv 4 \pmod 5$
$126 \equiv 4 \pmod 5$
$168 \equiv 3 \pmod 5$
$210 \equiv 0 \pmod 5$

This means that one of these 5 values is divisible by 5, and isn't prime unless that value is indeed 5.

Let's try N = 215. These five values are 5, 47, 89, 131 and 173, all of which are prime!

For any value of N > 215, we're guaranteed that one of the 5 values is divisible by 5 and ALL of the values are greater than 5, hence the value on that list divisible by 5 won't be prime.

Thus, we've proved that **215** satisfies the requirements and that no larger integer could possibly do so.

2005-12) For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(n) = 1$ and $\tau(6) = 4$. Define S(n) by

$$S(n) = \tau(1) + \tau(2) + \cdots + \tau(n)$$

Let a denote the number of positive integers n ≤ 2005 with S(n) odd, and let b denote the number of positive integers n ≤ 2005 with S(n) even. Find |a – b|.

Solution (Ans = 025)
Recall that the only numbers with an odd number of positive integer divisors are perfect squares. (For all other numbers that aren't perfect squares, their divisors come in pairs (a, n/a) that multiply to n but aren't equal to each other.)

It follows that the parity of S(n) only changes when we transition from n – 1 to n, where n is a perfect square.

We have S(1), S(2), and S(3) are odd.
Then S(4), S(5), S(6), S(7) and S(8) are even, and so forth.
So, we just have to investigate what happens towards the end of our range.
Recall that $44^2 = 1936$ and $45^2 = 2025$.
Thus, $S(43^2)$, … $S(44^2 – 1)$ are all odd and
S(1936), S(1937), …, S(2005) are all even.

Let's just add the odd ones up.

$S(43^2)$, … $S(44^2 – 1)$ is a set of 87 terms (from S(1849) to S(1935).

Notice that the set of odd terms come groups sized 3, 7, 11, …, 87. The difference between successive perfect squares are successive odd numbers, so the sizes of each group with the same parity will differ by 4. The sum of this arithmetic series is $\frac{(3+87)}{2} \times 22 = 990$. It follows that there are 2005 – 990 = 1015 even terms and the desired value of |a – b| = |990 – 1015| = 25.

Alternatively, we could notice that taking the difference # of odd terms minus # even terms would be the sum (3 – 5) + (7 – 9) … + (83 – 85) + 87 – 70 = -2(21) + 17 = -42 + 17 = -25

Problems from recent AIMEs

2020-I-4) Let S be the set of positive integers N with the property that the last four digits of N are 2020, and when the last four digits are removed, the result is a divisor of N. For example, 42,020 is in S because 4 is a divisor of 42,020. Find the sum of all the digits of all the numbers in S. For example, the number 42,020 contributes 4 + 2 + 0 + 2 + 0 = 8 to this total.

Solution (Ans = 093)
Let N = 10000x + 2020. We are given that x | N.
2020 = N – 10000x
Since x | (N – 10000x), it follows that x | 2020.

Thus, the question is asking us to find the sum of the digits in all of the divisors of 2020, with the number 2020 appended to the end of it.

2020 = 10 x 202 = 2 x 5 x 2 x 101 = 2^2 x 5 x 101

This number has 3 x 2 x 2 = 12 divisors. Let's just write them down:

1	2	4	5	10	20	Sum Digits = 15
101	202	404	505	1010	2020	Sum Digits = 30

There are 12 2020s to account for with a sum of digits of 12 x 4 = 48.

That means our final answer is 45 + 48 = **93.**

2022-I-7) Let a, b, c, d, e, f, g, h and i be distinct integers from 1 to 9. The minimum possible positive value of

$$\frac{a \times b \times c - d \times e \times f}{g \times h \times i}$$

can be written as m/n, where m and n are relatively prime positive integers. Find m + n.

Solution (Ans = 289)
We want to maximize the denominator while bringing the numerator as close to 0 as possible (1 would be ideal).

Let's think about getting a numerator of 1. This required two consecutive integers, in factorized form to be expressible with distinct digits. In order to get a numerator of 1, one term must be odd and the other even. The even term must be comprised of 2, 4, 6 and 8. The possibilities here are 36, 64, 96, 192. The matching possibilities would be (35 or 37), (63 or 65), (95 or 97), (191 or 193). Off hand, most of these 8 options are impossible, but let's go through them from smallest to largest. Let's try 35 first. 35 = 1 x 5 x 7, so we have a match! This leads us to the solution

$$\frac{2 \times 3 \times 6 - 1 \times 5 \times 7}{4 \times 8 \times 9} = \frac{1}{288}$$

We can verify that there's no smaller set of digits to get a numerator of 1, so if the correct answer has a numerator of 1, this is it.

Now, let's just make sure no answer with a numerator of 2 is smaller. The maximum possible denominator before reducing would be 7 x 8 x 9. This is $7/4^{th}$ of 3 x 8 x 9. Since 7/4 < 2, it's impossible for any answer with a denominator of 2 to be a smaller fraction that this one.

Thus, the answer is 1 + 288 = **<u>289.</u>**

2023 AIME-I 4) The sum of all positive integers m such that $\frac{13!}{m}$ is a perfect square can be written as $2^a 3^b 5^c 7^d 11^e 13^f$, where a, b, c, d, e and f are positive integers. Find a + b + c + d + e + f.

<u>Solution (Ans = 012)</u>
$13! = 2^{10}3^5 5^2 7^1 11^1 13^1$. (Note: We can use the algorithm to repeatedly cancel copies of a prime to see how many times each prime divides into 13! For 2, we have 13/2 + 6/2 + 3/2 = 6 + 3 + 1 = 10.)

All integers m that satisfy the query will be of the form $2^{2a'}3^{2b'+1}5^{2c'}7^1 11^1 13^1$.

The ranges for a', b' and c' are: $0 \leq a' \leq 5$, $0 \leq b' \leq 2$, $0 \leq c' \leq 1$, Using the fact that we can list the sum of all integers that fit this form as a product of terms where each term has the appropriate prime to different powers, we desire to find the largest prime number in the product:

$(2^0 + 2^2 + 2^4 + 2^6 + 2^8 + 2^{10})(3^1 + 3^3 + 3^5)(5^0+5^2)(7)(11)(13)$

The values of the first three terms are:

1024 + 256 + 64 + 16 + 4 + 1 = 1365 = 5 x 273 = 5 x 3 x 7 x 13
243 + 27 + 3 = 273 = 3 x 91 = 3 x 7 x 13
1 + 25 = 26 = 2 x 13

It follows that the desired product is

(5 x 3 x 7 x 13) x (3 x 7 x 13) x (2 x 13) x 7 x 11 x 13 = $2^1 3^2 5^1 7^3 11^1 13^4$

It follows that a + b + c + d + e + f = 1 + 2 + 1 + 3 + 1 + 4 = **12.**

2023 AIME-II 5) Let S be the set of all positive rational numbers r such that when the two numbers r and 55r are written as fractions in lowest terms, the sum of the numerator and denominator of one fraction is the same as the sum of the numerator and denominator of the other fraction. The sum of all the elements of S can be expressed in the form p/q, where p and q are relatively prime positive integers. Find p + q.

Solution (Ans = 719)

Let r = p/q so we have:

$\frac{p}{q}$ and $\frac{55p}{q}$

Without loss of generality, we may assume that gcd(p, q) = 1 since for both fractions any common divisor will divide out. Thus, it's clear that in order for the numerator and denominator to sum to the same thing when the fractions are reduced to lowest terms, that q must share a divisor with 55.

It follows that q = 5x or q = 11y. Let's try each of these substitutions

$\frac{p}{5x}$ and $\frac{55p}{5x} = \frac{11p}{x}$, this gives us p + 5x = 11p + x → 4x = 10p → 2x = 5p

Plugging in potential values we the solution x = 5, p = 2. Notice that gcd(x, p) = 1 because otherwise the fractions wouldn't be in lowest terms. So this solution leads to the fraction $r = \frac{2}{25}$.

Now, let's try q = 11y:

$\frac{p}{11y}$ and $\frac{55p}{11y} = \frac{5p}{y}$, this gives us p + 11y = 5p + y → 10y = 4p → 5y = 2p. Since gcd(p, y) = 1, we must have y = 2, p = 5, which leads to the solution, $r = \frac{5}{22}$.

These are the only solutions, so our desired sum is $\frac{2}{25} + \frac{5}{22} = \frac{2(22)+5(25)}{550} = \frac{44+125}{550} = \frac{169}{550}$.

The final answer is 169 + 550 = **719.**