Probability – AIME Preparation OMC 1/10/2024

1. Two counting problems
2. Binomial Distribution
3. Recursive Formulas and building probability via state
4. Probability Trees, Conditional Probability

Problems to go with Lecture Topics (from Old AIMEs)

1984-11) A gardener plans 3 maple trees, 4 oak trees and 5 birch trees in a row. He plans them in random order, each arrangement being equally likely. Let m/n in lowest terms be the probability that no two birch trees are next to one another. Find m + n.

1988-5) Let m/n, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}.

1989-5) When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let i/j, in lowest terms, be the probability that the coin comes up heads exactly 3 times out of 5. Find i + j.

1990-9) A fair coin is to be tossed ten times. Let i/j, in lowest terms, be the probability that heads never occurs on consecutive tosses. Find i + j.

Problems from recent AIMEs

2022 AIME-II 2) Azar, Carl, John and Sergey are the four players left in a singles tennis tournament. They are randomly assigned opponents in the semifinal matches, and the winners of those matches play each other in the final match to determine the winner of the tournament.

When Azar plays Carl, Azar will win the match with probability 2/3. When either Azar or Carl plays either Jon or Sergey, Azar or Carl will win the match with probability ¾. Assume that the outcomes of different matches are independent. The probability that Carl will win the tournament is p/q, where p and q are relatively prime positive integers. Find p + q.

2021 AIME-II 8) An ant makes a sequence of moves on a cube where a move consists of walking from one vertex to an adjacent vertex along an edge of the cube. Initially, the ant is at the vertex of the bottom face of the cube and chooses one of the three adjacent vertices to move to as its first move. For all moves after the first move, the and does not return to its previous vertex, but chooses to move to one of the other two adjacent vertices. All choices are selected at random so that each of the possible moves is equally likely. The probability that after exactly 8 moves that the ant is at a vertex of the top face on the cube is m/n, where m and n are relatively prime positive integers. Find m + n.

2020 AIME-I 9) Let S be the set of positive integer divisors of 20^9. Three numbers are chosen independently and at random with replacement from the set S and labeled a_1, a_2 and a_3 in the order they are chosen. The probability that both a_1 divides a_2 and a_2 divides a_3 is m/n, where m and n are relatively prime positive integers. Find m.

2021 AIME-I 12) Let $A_1A_2...A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. The expected number of minutes until the frogs stop jumping is m/n, where m and n are relatively prime positive integers. Find m + n.