Counting/Combinatorics – AIME Preparation OMC Homework for 1/3/2024

Problems from Old AIMEs

1997-2) The nine horizontal and nine vertical lines on a 8 x 8 checker board form *r* rectangles, of which *s* are squares. The number *r*/*s* can be written in the form *m*/*n*, where *m* and *n* are relatively prime positive integers. Find *m* + *n*.

Solution (Ans = 125)

We can choose any 2 of the horizontal lines in $\binom{9}{2} = 36$ ways to form the horizontal lines of a rectangle. We can similarly choose any 2 of the vertical lines in 36 ways to form the vertical lines of the rectangle. Thus, r = 36 x 36 = 1296. Now, let's count all of the squares. There are 64 1 x 1 squares, 49 2 x 2 squares, 36 3 x 3 squares, etc. Thus, the total number of squares is

$$\sum_{i=1}^{8} i^2 = \frac{8(8+1)(2\times 8+1)}{6} = 12 \times 17 = 204$$

Finally $\frac{r}{s} = \frac{204}{1296} = \frac{51}{324} = \frac{17}{108}$.

Thus, m + n = 17 + 108 = **125.**

1993-11) Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner of a game is the first person to obtain a head. Alfred and Bonnie play this game several times with the stipulation that the loser of a game goes first in the next game. Suppose that Alfred goes first in the first game, and that the probability that he wins the sixth game is m/n. where m and n are relatively prime positive integers. What are the last three digits of m + n?

Solution (Ans = 093)

If a person goes first, let their probability of winning be X. Then X satisfies the following equation:

$$X = \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times X$$

Because either the person going first wins on their first turn (with probability ½), OR they don't get a heads and their opponent doesn't get a heads (both probability ½), at which point the game has returned to its beginning state, so the probability player 1 wins at this time is X again.

Solving, we find that $\frac{3}{4}X = \frac{1}{2}$, so $X = \frac{2}{3}$.

Let p(n) be the probability that Alfred wins game n. It follows that 1 – p(n) is the probability that Bonnie wins game n.

Now, let's derive a recursive formula for p(n) for n > 1. Either Alfred won the previous game or Bonnie did. If Alfred won the previous game (which occurs p(n-1) of the time), then Bonnie starts game n and Alfred's chance of winning it is 1/3. Alternatively, if Bonnie won the previous game (which occurs 1 – p(n-1) of the time), then Alfred starts game n and his chance of winning it is 2/3. This gives us the following formula:

$$p(n) = \frac{1}{3}p(n-1) + \frac{2}{3}(1 - p(n-1))$$

$$p(n) = \frac{1}{3}p(n-1) + \frac{2}{3} - \frac{2}{3}p(n-1)$$

$$p(n) = \frac{2}{3} - \frac{1}{3}p(n-1)$$

We know that p(1) = 2/3, so we can build up the answers as follows:

$$p(2) = \frac{2}{3} - \frac{1}{3} \times \frac{2}{3} = \frac{6}{9} - \frac{2}{9} = \frac{4}{9}$$

$$p(3) = \frac{2}{3} - \frac{1}{3} \times \frac{4}{9} = \frac{18}{27} - \frac{4}{27} = \frac{14}{27}$$

$$p(4) = \frac{2}{3} - \frac{1}{3} \times \frac{14}{27} = \frac{54}{81} - \frac{14}{81} = \frac{40}{81}$$

$$p(5) = \frac{2}{3} - \frac{1}{3} \times \frac{40}{81} = \frac{162}{243} - \frac{40}{243} = \frac{122}{243}$$

$$p(6) = \frac{2}{3} - \frac{1}{3} \times \frac{122}{243} = \frac{486}{729} - \frac{122}{729} = \frac{364}{729}$$

364 + 729 = 1093, so the last three digits are **093.**

There's definitely a pattern here where we flip flop between barely greater than ½ and barely below ½, which can be proven via induction to provide a closed-form formula for p(n).

1994-9) A solitaire game is played as follows. Six distinct pairs of matched tiles are placed in a bag. The player randomly draws tiles one at a time from the bag and retains them, except that matching tiles are put aside as soon as they appear in the player's hand. The game ends if the player ever holds three tiles, no two of which match; otherwise the drawing continues until the bag is empty. The probability that the bag will be emptied is p/q, where p and q are relatively prime positive integers. Find $p + q$.

Solution (Ans = 394)

Let p(n, k) = the answer to the question for n pairs of tiles, given that we are holding k unique tiles right now.

Let's think about p(n, 0) recursively.

On our first draw, we pick some tile with probability 1.
With probability $\frac{1}{2n-1}$ we pick the matching tile right afterwards and will succeed with probability p(n – 1, 0).

With probability $\frac{2n-2}{2n-1}$, we'll have two different tiles in our hand. At this point, the only way to win would be to match one of the two tiles in our hand, which occurs with probability $\frac{2}{2n-2}$. Thus, this gives us the recursive equation:

$$p(n,0) = \frac{1}{2n-1}p(n-1,0) + \frac{2n-2}{2n-1} \times \frac{2}{2n-2}p(n-1,1)$$

$$p(n,0) = \frac{1}{2n-1}p(n-1,0) + \frac{2}{2n-1}p(n-1,1)$$

Now, we must develop a recursive formula for p(n, 1). In this case, there are 2n – 1 other tiles not yet selected and you are holding 1 tile, with the matching one not yet selected.

With probability $\frac{1}{2n-1}$, you pick the matching tile and go to the state p(n – 1, 0). And again, with probability $\frac{2}{2n-1}$ you get to the state p(n – 1, 1):

$$p(n,1) = \frac{1}{2n-1}p(n-1,0) + \frac{2}{2n-1}p(n-1,1)$$

Note that p(n, 0) = p(n, 1). This actually makes a lot of sense because when you draw a tile with nothing, you always move to the state with 1 tile in your hand. So, redefine our function to be in terms of one parameter:

$$p(n) = \frac{3}{2n-1} \times p(n-1).$$

Our initial conditions are p(1) = p(2) = 1.

$$p(3) = \frac{3}{5} \times p(2) = \frac{3}{5}$$
$$p(4) = \frac{3}{7} \times p(3) = \frac{3}{7} \times \frac{3}{5} = \frac{9}{35}$$
$$p(5) = \frac{3}{9} \times p(4) = \frac{1}{3} \times \frac{9}{35} = \frac{3}{35}$$

$$p(6) = \frac{3}{11} \times p(5) = \frac{3}{11} \times \frac{3}{35} = \frac{9}{385}$$

Thus, the final answer is 9 + 385 = **394.**

1995-15) The *p* be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Given that *p* can be written in the form *m*/*n*, where *m* and *n* are relatively prime positive integers, find *m* + *n*.

Solution (Ans = 037)
The AoPS solution is much shorter than mine with only 2 unique variables, but I am still including my solution which utilizes the recursive problem formulation I taught.

Let p(h, t) equal the probability of achieving the goal given that we have ended on a string of h heads or t tails, where either h = 0 or t = 0, since the last streak has to be of one or the other.

Our final answer is $\frac{1}{2}p(1,0) + \frac{1}{2}p(0,1)$. But, the latter term is fairly easy to simplify because if we previously flip a tail, we'll lose if we flip another, and with probability ½, we will return to the state p(1, 0). Thus, $p(0,1) = \frac{1}{2}p(1,0)$. It follows that the final answer we seek is is $\frac{3}{4}p(1,0)$.

Let's work on figuring out p(1, 0):

$$p(1,0) = \frac{1}{2}p(2,0) + \frac{1}{4}p(1,0)$$

The idea here is that we could get 1 H to get to the state p(2, 0) or if we get a Tail, we must immediately flip a Head to get back to p(1, 0). The probability of a T followed by a head is ¼. Solve this for p(1, 0):

$$\frac{3}{4}p(1,0) = \frac{1}{2}p(2,0) \rightarrow p(1,0) = \frac{2}{3}p(2,0)$$

Now, let's solve for p(2, 0):

$$p(2,0) = \frac{1}{2}p(3,0) + \frac{1}{4}p(1,0) = \frac{1}{2}p(3,0) + \frac{1}{4} \times \frac{2}{3}p(2,0)$$

Notice that we get back to p(1,0), but we previously found a relationship between p(1, 0) and p(2, 0) so we can substitute that in. Now solve this for p(2, 0):

$$\frac{5}{6}p(2,0) = \frac{1}{2}p(3,0) \rightarrow p(2,0) = \frac{3}{5}p(3,0)$$

Also, note that $p(1,0) = \frac{2}{3} \times \frac{3}{5}p(3,0) = \frac{2}{5}p(3,0)$, by substituting in the previous result.

Now, let's solve for p(3, 0):

$$p(3,0) = \frac{1}{2}p(4,0) + \frac{1}{4}p(1,0) = \frac{1}{2}p(4,0) + \frac{1}{4} \times \frac{2}{5}p(3,0)$$

Simplifying we get:

$$\frac{9}{10}p(3,0) = \frac{1}{2}p(4,0) \rightarrow p(3,0) = \frac{5}{9}p(4,0)$$

Also note that $p(1,0) = \frac{2}{5} \times \frac{5}{9}p(4,0) = \frac{2}{9}p(4,0)$.

Finally, let's solve for p(4, 0):

$$p(4,0) = \frac{1}{2} + \frac{1}{4}p(1,0) = \frac{1}{2} + \frac{1}{4} \times \frac{2}{9}p(4,0) = \frac{1}{2} + \frac{1}{18}p(4,0)$$

Gather the p(4, 0) terms to get $\frac{17}{18}p(4,0) = \frac{1}{2} \rightarrow p(4,0) = \frac{9}{17}$.

It follows that the final result is $\frac{3}{4}p(1,0) = \frac{3}{4} \times \frac{2}{9} \times p(4,0) = \frac{1}{6} \times \frac{9}{17} = \frac{3}{34}$.

It follows that the final answer is 3 + 34 = **37.**

Problems from recent AIMEs

2021-I-1) Zou and Chou are practicing their 100 meter sprints by running 6 races against each other. Zou wins the first race, and after that, the probability that one of them wins a race is 2/3 is they won the previous race, but only 1/3 if they lost the previous race. The probability that Zou will win exactly 5 of 6 race six m/n, where m and n are relatively prime positive integers. Find m + n.

Solution (Ans = 097)

There are 5 different ways that Zou could win five races: ZCZZZZ, ZZCZZZ, ZZZCZZ, ZZZZCZ and ZZZZZC. For the first four options, the probability of the first event is 1 (we were given that Zou won the first race), and of the remaining 5 events, the probability of three of them is 2/3 (same winner as previous winner) and the probability of the other two events is 1/3. For the last option, there's only 1 switch in winner, so the probability of four of the events is 2/3 and one of them is 1/3. Adding, we get:

$$4 \times (\frac{2}{3})^3 \times (\frac{1}{3})^2 + (\frac{2}{3})^4 \times (\frac{1}{3})^1 = \frac{4 \times 2^3 + 2^4}{243} = \frac{48}{243} = \frac{16}{81}$$

The desired answer is 16 + 81 = **97.**

2022-I-9) Elina has twelve blocks, two each of red (R), blue (B), yellow (Y), green (G), orange (O), and purple (P). Call an arrangement of blocks ***even*** if there is an even number of blocks between each pair of blocks of the same color. For example, the arrangement

R B B Y G G Y R O P P O

Is even. Elina arranges her blocks in a row in random order. The probability her arrangement is even is m/n, where m and n are relatively prime positive integers. Find m + n.

Solution (Ans = 247)

The key observation here is that if 2 blocks have an even number of blocks between them, the ***parity*** of the indexes they are in (order from left) is different. Thus, for each pair of blocks, one must be on an odd index and the other an even index. Thus, we can place one R, B, Y, G, O and P in locations 1, 3, 5, 7, 9 and 11 in exactly 6! ways, and then do the same with the other copies of the letters in the locations 2, 4, 6, 8, 10 and 12 in 6! Ways.

The sample space is the total number of arrangements of the 12 letters which is $\frac{12!}{2^6}$. Thus, the desired answer is:

$$\frac{6!\,6!}{\frac{12!}{2^6}} = \frac{2^6(6!)}{7 \times 8 \times 9 \times 10 \times 11 \times 12} = \frac{64 \times 6}{7 \times 8 \times 9 \times 11} = \frac{16}{7 \times 3 \times 11} = \frac{16}{231}$$

The desired sum is 16 + 231 = **247.**

2023-I-1) Five men and nine women stand equally spaced around a circle in random order. The probability that every man stands diametrically opposite a woman is m/n, where m and n are relatively prime positive integers. Find m + n.

Solution (Ans = 191)

We can use multiplication principle here. Imagine that each man is distinguishable. (It doesn't change the answer either way.) Our sample space is then 14 x 13 x 12 x 11 x 10, since for each man we have one fewer slot to choose after placing the previous men.

Now, let's count how many of these ways work.

The first man can be successfully placed in 14 ways since no one else is placed. But the second man can't be placed diametrically opposite to the first, so he has only 12 valid locations to go. Similarly, given that the first 2 men aren't opposite each other, the third man has only 10 valid locations to go (2 open ones are opposite the two men previously placed). The fourth man will have 8 possible locations and the last man will have 6 possible locations. Thus, the desired probability is:

$$\frac{14 \times 12 \times 10 \times 8 \times 6}{14 \times 13 \times 12 \times 11 \times 10} = \frac{48}{143}$$

It follows that the desired answer is 48 + 143 = **191.**

2023-I-6) Alice knows that 3 red cards and 3 black cards will be revealed to her one at a time in random order. Before each card is revealed, Alice must guess its color. If Alice plays optimally, the expected number of cards she will guess correctly is m/n, where m and n are relatively prime positive integers. Find m + n.

Solution (Ans = 051)

Let E(r, b) equal the expected number of cards Alice guesses correctly given that she has r red cards left and b black cards left. Let's build up the table:

E(r, 0) = r, E(0, b) = b, also note that E(r, b) = E(b, r).

$E(1,1) = \frac{1}{2} \times (1 + E(1,0)) + \frac{1}{2}E(0,1) = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$, basically, half the time we guess 1 correct when our first guess is wrong and the other half of the time we guess both correct.

$E(2,1) = \frac{2}{3}(1 + E(1,1)) + \frac{1}{3}E(2,0) = \frac{2}{3}\left(1 + \frac{3}{2}\right) + \frac{1}{3} \times 2 = \frac{7}{3}$

Above, with probability 2/3 we guess the more probable item correctly and with probability 1/3 we get that first guess wrong (but then the last two correct).

$$E(2,2) = \frac{1}{2}(1 + E(2,1)) + \frac{1}{2}E(2,1) = \frac{1}{2}(1 + 2E(2,1)) = \frac{1}{2}\left(1 + \frac{14}{3}\right) = \frac{17}{6}$$

We are almost there! Now, let's work out E(3, 1):

$$E(3,1) = \frac{3}{4}(1 + E(2,1)) + \frac{1}{4}E(3,0) = \frac{3}{4}\left(1 + \frac{7}{3}\right) + \frac{3}{4} = \frac{13}{4}$$

And E(3, 2):

$$E(3,2) = \frac{3}{5}(1 + E(2,2)) + \frac{2}{5}E(3,1) = \frac{3}{5}\left(1 + \frac{17}{6}\right) + \frac{2}{5} \times \frac{13}{4} = \frac{23}{10} + \frac{13}{10} = \frac{18}{5}$$

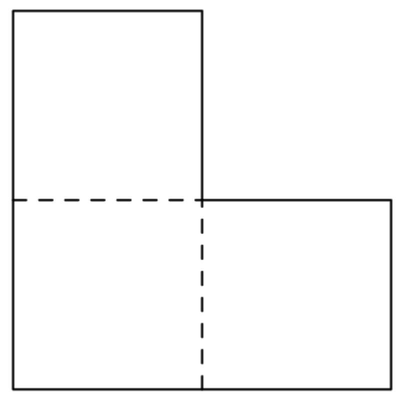
Finally, notice that for E(3, 3), we add ½ to our expected value on the first guess, and from there we are guaranteed to be in a state equivalent to E(3, 2). Thus, our final answer is:

$$\frac{1}{2} + \frac{18}{5} = \frac{41}{10}$$

It follows that the desired answer is 41 + 10 = **51.**

Note: One can draw out a probability tree and add up the probability for each branch as well. If we use symmetry of the (3, 3), (2, 2) and (1, 1) cases, there are five separate branches to add up.

2023-II-6) Consider the L-shaped region formed by three unit squares joined at their sides, as shown below. Two points A and B are chosen independently and uniformly at random from inside the region. The probability that the midpoint of $\overline{AB}$ also lies inside this L-shaped region can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution (Ans = 035)

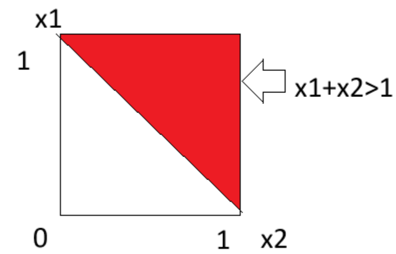
Let's calculate the opposite, the probability the midpoint of the segment is outside the given figure. The only way this could happen is if one point is in the upper square and the other point is in the right square. The probability this occurs is $2 \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{9}$. (Either pt A is on the top and B is on the right of vice versa, so we multiply with 2. The probability of each event is 1/3 since the whole area is 3 and each square has area 1.))

Now, let the coordinates of the bottom left by (0, 0). Then a point in the top left can be expressed as (x1, y1 + 1) and a point in the bottom right can be expressed as (x2 + 1, y2). The midpoint of the segment connecting these two points is:

$$(\frac{x1 + x2 + 1}{2}, \frac{y1 + y2 + 1}{2})$$

Note that we have $0 \leq$ x1, x2, y1, y2 ≤ 1. In order for the midpoint to be outside the diagram we want $\frac{x1+x2+1}{2} > 1$, so $x1 + x2 > 1$ and $\frac{y1+y2+1}{2} > 1$, so $y1 + y2 > 1$.

Since both values of x and y are equally distributed between 0 and 1, the probability that their sum is greater than 1 is just ½. (We can visualize this in two dimensions as a square with x1 in one dimension and x2 in the other, and the area above the line y = 1 – x, a right triangle with legs both side 1 is when x1 + x2 > 1.)



Thus, the probability the midpoint is outside the L-shaped design is $\frac{2}{9} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{18}$.

It follows that the probability the midpoint is in the design is $1 - \frac{1}{18} = \frac{17}{18}$. The required answer is 17 + 18 = **35.**

2024-I-4) Jen enters a lottery by picking 4 distinct numbers from S={1,2,3,…,9,10}. Four numbers are randomly chosen from S. She wins a prize if at least two of her numbers were 2 of the randomly chosen numbers, and wins the grand prize if all four of her numbers were the randomly chosen numbers. The probability of her winning the grand prize given that she won a prize is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

Solution (Ans = 116)

The sample space is $\binom{10}{4} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = \frac{10\times9\times7}{3} = 210$.

The number of ways she wins no prize is if she picks 4 wrong numbers, which can be done in $\binom{6}{4} = 15$ ways or if she chooses 3 wrong numbers and one correct number, which can be done in $\binom{6}{3}\binom{4}{1} = 20 \times 4 = 80$. There are 95 ways she loses. It follows that she wins a prize in 210 – 95 = 115 ways.

Her probability of winning the grand prize, given that she won a prize is $\frac{1}{115}$, since only 1 of those 115 ways of winning a prize get her the grand prize. It follows that m + n = 1 + 115 = **116.**

2024-I-11) Each vertex of a regular octagon is independently colored either red or blue with equal probability. The probability that the octagon can then be rotated so that all of the blue vertices end up at positions where there were originally red vertices is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

Solution (Ans = 371)

The question is asking how many 8 letter strings where each letter is R or B exist such that at least one of the cyclic rotations of the string place each B in a location where there were previously Bs. Thus, our sample space is $2^8 = 256$.

If there are 0 vertices colored blue, we are good.
If there is 1 vertex colored blue, we are good (rotate 1).
If there are 2 vertices colored blue, if they are consecutive, rotate by 2, otherwise, rotating by 1 definitely works.

If there are 3 vertices colored blue. We have some more cases to go through:

BBBRRRRR (3 Bs together) – okay
BBRBRRRR (3 Bs out of any 4) – okay, rotate 4
BBRRBRRR (3 Bs not out of any 4) – okay, rotate 2, notice that B can't be in the last 2 slots.
BRRBRBRR (no Bs consecutive) – okay, rotate 1

So all of these cases work as well!

Now, the hardest case is 4Bs and 4 Rs. These only work if the 4 Bs can rotate exactly to where the 4 Rs are. Let's go through these cases, starting with B's, and counting how many unique rotations each one creates:

BRBRBRBR (2 rotations, since rotate 2 is the same thing)
BBRRBBRR (4 rotations, since rotate 4 is the same thing)
BBBBRRRR (8 rotations, since each of the 8 rotations has a B in a different place)

These are probably the easiest ones to see since they are cyclic, so to speak. Here is the last one that works:

BBRBRRBR (8 rotations since none are identical)

This one is possible with a rotation of 4.

Let's add up all the settings that work:

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + 2 + 4 + 8 + 8 = 1 + 8 + 28 + 56 + 22 = 115$$

The desired probability is $\frac{115}{256}$. The desired answer is 115 + 256 = **371.**