<u>Counting/Combinatorics – AIME Preparation OMC Homework for 1/3/2024</u>

<u>Problems from 102 Combinatorial Problems</u>

10) Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will 20! be the resulting product?

<u>Solution (Ans = 128)</u>
The prime factorization of 20! is of the form $2^a3^b5^c7^d11^e13^f17^g19^h$, where a, b, c, d, e, f, g and h are each positive integers. (These values can be calculated but doing so is unnecessary to solve the problem.) This is because the product the integers from 1 to 20 only contains all prime numbers in between 1 and 20, inclusive. If a fraction is in lowest terms, then any prime factor that appears in the numerator can not appear in the denominator. Thus, for each of terms in the prime factorization of 20! we may choose to place it either in the numerator or denominator. Since we have 2 choices for each of 8 terms, that is a possible 2^8 = 256 fractions. However, some of these fractions might be greater than 1. Luckily, for any arbitrary fraction $\frac{x}{y}$ in the list of 256, the fraction $\frac{y}{x}$ must also be in the list. Exactly one of the two fractions is less than 1. It follows that we can get our final answer for number of fractions in between 0 and 1 by dividing the total number of fractions by 2. Thus, the final answer is 256/2 = **128.**

15) For how many pairs of consecutive integers in the set {1000, 1001, 1002, …, 2000} is no carrying required when the two integers are added?

<u>Solution (Ans = 156)</u>
Let the first integer in the pair be 1abc, so we are adding 1abc and 1ab[c+1].
c = 0, 1, 2, 3, 4, or 9 since all other values would cause a carry. Our cases look like:

1ab0	1ab1	1ab2	1ab3	1ab4	1ab9
1ab1	1ab2	1ab3	1ab4	1ab5	1a[b+1]0

In the first five cases, it's clear that we don't get a carry if and only if 0 ≤ a, b ≤ 4. Thus, if c is one of 0, 1, 2, 3, or 4, then there are also 5 choices for a and 5 choices for b. This results in 5 x 5 x 5 = 125 choices.

Now we must count the number of cases highlighted in yellow above that have no carries. If b = 0, 1, 2, 3, or 4, then a must also be 0, 1, 2, 3, or 4. Thus this is an additional 5 x 5 = 25 cases.

The only cases we haven't counted are of the form 1a99 + 1[a+1]00. There are six possible values for a that work here: a = 0, 1, 2, 3, 4, or 9.

Thus, the final answer is 125 + 25 + 5 + 1 = **156.**

16) Nine chairs in a row are to be occupied by six students and professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

Solution (Ans = 60)

Let the students be separators between the professors, since the professors may NOT sit next to each other. Put blanks between the students, but not on the ends to represent where the professors may sit:

S1 ___ S2 ___ S3 ___ S4 ___ S5 ___ S6

We can have Alpha sit in one of 5 seats, then Beta can sit on one of the four remaining seats, and finally Gamma can sit in one of the remaining 3 seats. Thus, the final answer is 5 x 4 x 3 = **60.** (Note: Since the professors have names, they are distinguishable.)

20) Two of the squares of a 7 x 7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

Solution (Ans = 300)

We can choose 2 squares out of 49 in $\binom{49}{2} = \frac{49 \times 48}{2} = 49 \times 24 = 1176$ ways.
But, some of these 1176 ways are rotations of the same design. In particular, some choices are counted twice while others are counted 4 times. A typical case is counted four times, since there are four 90 degree rotations. But for some figures, rotating it by 90 degrees twice will lead to the same design, so for this case, of the four rotations, only 2 of them were counted in the 1176. Let x be the number of designs that have four unique rotations and y be the number of designs with two unique rotations. It follows that 4x + 2y = 1176.

Let's calculate y. These are the symmetric designs that look the same when rotated 180 degrees. This is not possible if we pick the middle square. Consider choosing an arbitrary non-middle square. There is a unique symmetric matching square that is where this square is would rotate under a 180 degree rotation. If this is our second square, then this pair of squares counts as part of our value of y. There are 24 squares we could pick as our first square (forcing it to be in the top three rows or fourth row and first three columns), so a total of 24 total pairs that are symmetrical. Thus, y = 12 and our equation simplifies to:

4x + 2(12) = 1176
4x = 1176 – 24
4x = 1152
x = 288

Our final answer is x + y = 288 + 12 = **300.**

30) Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

Solution (Ans = 589)
$n^2 = 2^{62}3^{38}$ and has (62 + 1)(38 + 1) = 63 x 39 = 2457 divisors, one of which is n. Thus, there are a total of 2456 divisors of n^2 which are either greater than n or less than n. In fact, these divisors come in pairs that multiply to n^2, so exactly half of them are less than n and the other half are greater than n. Thus, the number of divisors of n^2 that are less than n is 2456/2 = 1228.

The total number of divisors of n is (31 + 1)(19 + 1) = 32 x 20 = 640. Of these, 639 are less than n.

It follows that the number of divisors of n^2 that aren't divisors of n but are less than it is equal to 1228 – 639 = **589.**

Problems from recent AIMEs

2022-I-6) Find the number of ordered pairs of integers (a, b) such that the sequence

3, 4, 5, a, b, 30, 40, 50

is strictly increasing and no set of four (not necessarily consecutive) terms forms an arithmetic progression.

Solution (Ans = 228)
Neither a nor b can equal 6 or 20, thus a and b must be integers chosen from the set {7, 8, …,19, 21, 22, 23, …, 29}. Thus there are a total of $\binom{22}{2} = \frac{22\times21}{2} = 11 \times 21 = 231$ possible choices for a and b. Now, let's see how many of these choices are invalid.

Of the known values in the sequence, we've already removed situations where 3 of known values were in the arithmetic progression of four terms. So, now, we must look for situations where a and b are both part of the arithmetic progression of four terms as well as two other numbers. Let's consider our choices (only choices that don't have 6 or 20 for either a or b):

3, 5 → a = 7, b = 9 (subtract this)
3, 30 → a = 12, b = 21 (subtract this)
4, 40 → a = 16, b = 28 (subtract this)

Thus, our final answer is 231 – 3 = **228.**

2022-II-5) Twenty distinct points are marked on a circle labeled 1 through 20 in clockwise order. A line segment is drawn between every pair of points whose labels differ by a prime number. Find the number of triangles formed whose vertices are among the original 20 points.

<u>Solution (Ans = 72)</u>

Let the points for the triangle be a, b and c with a < b < c. Then b – a is a prime number and c – b is a prime number, as well as c – a. We know that c – a must be odd. But c – a = (b – a) + (c – b). The only way the sum of two prime numbers is an odd prime is if one of the two primes is 2. Thus, either b – a = 2 or c – b = 2. Without loss of generality, assume b – a = 2, and then we can multiply our result by 2 to get the final result. Let's count the number of solutions where b – a = 2:

(a, b) = (1, 3) → c = [6, 8, 14, 20] → 4 solutions
(a, b) = (2, 4) → c = [7, 9, 15] → 6 x 3 = 18 solutions
(3, 5) → c = [8, 10, 16]
…
(7, 9) → c = [12, 14, 20]
(a, b) = (8, 10) → c = [13, 15] → 6 x 2 = 12 solutions
…
(13, 15) → c = [18, 20]
(a, b) = (14, 16) → c = [19] → 2 x 1 = 2 solutions
(15, 17) → c = [20]

Totals with b – a = 2 is 4 + 18 + 12 = 2 = 36.
Similarly, there are 36 solutions with c – b = 2.
Thus, there are a total of 2 x 36 = **<u>72</u>** solutions.

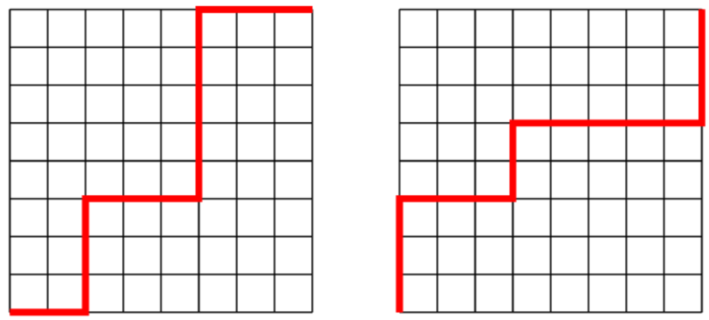
2023-I-3) A plane contains 40 lines, no two of which are parallel. Suppose that there are 3 points where exactly 3 lines intersect, 4 points where exactly 4 points intersect, 5 points where exactly 5 points intersect, 6 points where exactly 6 points intersect, and no points where more than 6 lines intersect. Find the number of points where exactly 2 lines intersect.

<u>Solution (Ans = 607)</u>

There are a total of $\binom{40}{2} = \frac{40\times 39}{2} = 20 \times 39 = 780$ pairs of distinct lines. If 3 lines intersect at one point, then there is a count of 3 pairs of lines for that one point. If 4 lines intersect at one point, then there is a count of $\binom{4}{2} = 6$ pairs of lines that intersect at that point. If 5 lines intersect at one point, then there is a count of $\binom{5}{2} = 10$ pairs of lines that intersect at that point. If 6 lines intersect at one point, then there is a count of $\binom{6}{2} = 15$ pairs of lines that intersect at that point. Let x be the answer to the question. Then x satisfies the following equation:

$$x + 3 \times 3 + 4 \times 6 + 5 \times 10 + 6 \times 15 = 780$$
$$x + 9 + 24 + 50 + 90 = 780$$
$$x + 173 = 780$$
$$x = \mathbf{607}$$

2024-I-6) Consider the paths of length 16 that follow the lines from the lower left corner to the upper right corner on an 8×8 grid. Find the number of such paths that change direction exactly four times, like in the examples shown below.



Solution (Ans = 294)

We have two types of paths, ones which start out horizontally and others which start out vertically. The former have 3 horizontal segments and 2 vertical segments. The latter have 2 vertical segments and 3 horizontal segments. For each valid path of the former type, there is a different unique matching path of the latter type. So, let's count all the ways to do 3 horizontal segments and 2 vertical segments and multiply the result by 2.

Let a = distance moved horizontally first
Let b = distance moved vertically after first turn.
Let c = distance moved horizontally after second turn
Let d = distance moved vertically after third turn and
Let e = distance moved horizontally after the last turn

We have a + c + e = 8 and b + d = 8, with each being positive integers. Let a = a' + 1, so the restriction on a' is that it's a non-negative integer, and do similar substitutions for b, c, d and e. We then get the equations:

a' + 1 + c' + 1 + e' + 1 = 8 and b' + 1 + d' + 1 = 8
a' + c' + e' = 5 and b' + d' = 6

The first equation is combinations with repetition with n = 5 and r = 3. There are $\binom{5+2}{2} = 21$ solutions to the first equation. The second equation is also combinations with repetition with n = 6 and r = 2. There are $\binom{6+1}{1} = 7$ solutions to the second equation. Since we can pair up any solution to the first equation with any solution to the second equation, there are a total of 21 x 7 = 147 paths with 3 horizontal segments and 2 vertical segments. It follows that the final answer to the question is 147 x 2 = **294.**

2024-II-6) Alice chooses a set A of positive integers. Then Bob lists all finite nonempty sets B of positive integers with the property that the maximum element of B belongs to A. Bob's list has 2024 sets. Find the sum of the elements of A.

Solution (Ans = 55)

Let's say A has a value 8 in its set. Consider a set B which has 8 as its maximum element. If B can have any positive integers, there are 2^7 possible sets B could be, since for each integer, x, 1 through 7, inclusive, x is either in the set or not in the set.

It follows that if A's elements are {a1, a2, a3, a4, …, ak}, then the number of elements in B will equal $2^{a1-1} + 2^{a2-1} + 2^{a3-1} + \ldots 2^{ak-1}$. Thus, we seek the binary representation of 2024.

2	2024	
2	1012	R 0
2	506	R 0
2	253	R 0
2	126	R 1
2	63	R 0
2	31	R 1
2	15	R 1
2	7	R 1
2	3	R 1
2	1	R 1
2	0	R 1

Thus, $2024_{10} = 11111101000_2$.

It follows that the set A is {4, 6, 7, 8, 9, 10, 11}.

The desired sum is 4 + 6 + 7 + 8 + 9 + 10 + 11 = **<u>55.</u>**

2024-II-9) There are 25 indistinguishable white chips and 25 indistinguishable black chips, find the number of ways to place some of these chips in a 5 x 5 grid such that:

1. each cell contains at most one chip
2. all chips in the same row and all chips in the same column have the same color
3. any additional chip placed on the grid would violate one or more of the previous two Conditions

<u>Solution (Ans = 902)</u>
Let's consider an example where rows 2, 3 and 5 have black chips only and columns 3 and 4 have white chips only. Let's see if we can adhere to these restrictions and make it impossible to add any chips:

		W	W	
B	B			B
B	B			B
		W	W	
B	B			B

Once we banned black from columns 3 and 4, we were required to fill black in all other columns (1, 2 and 5), but only in the rows that were black rows. Similarly, once we banned white from rows 2, 3 and 5, we had to fill columns 3 and 4, but only in rows 1 and 4.

Perhaps a better explanation is that we can choose any non-empty proper subset of rows for black along with any non-empty proper subset of columns for black. Once we choose these, the choices for white's rows and columns are the complementary sets. Thus, there are $2^5 – 2= 30$ choices for black's rows and $2^5 – 2 = 30$ choices for black's columns, for a total of 30 x 30 = 900 possible arrangements of chips.

The only possibilities we didn't entertain are if black and all rows or white had all rows. In this case, to make it so no other chips could be placed, the entire grid has to be all black or all white. So there are 2 more possible solutions.

The final answer is 900 + 2 = **<u>902.</u>**