Questions about Polynomials and Roots

$f(x) = x^3 + ax^2 + bx + c = (x - r)(x - s)(x - t)$

Coefficient form Factored/Root Form

$x^3 + ax^2 + bx + c = x^3 - (r + s + t)x^2 + (rs + rt + st)x - rst$

a = -sum of roots

b = + product of roots by 2

c = - product of roots by 3

2023 AMC 10 A

Problem 21

Let $P(x)$ be the unique polynomial of minimal degree with the following properties:

- $P(x)$ has a leading coefficient 1,
- 1 is a root of $P(x) - 1$,
- 2 is a root of $P(x - 2)$,
- 3 is a root of $P(3x)$, and
- 4 is a root of $4P(x)$.

The roots of $P(x)$ are integers, with one exception. The root that is not an integer can be written as $\frac{m}{n}$, where m and n are relatively prime integers. What is $m + n$?

$\textbf{(A)}\ 41 \qquad \textbf{(B)}\ 43 \qquad \textbf{(C)}\ 45 \qquad \textbf{(D)}\ 47 \qquad \textbf{(E)}\ 49$

What do we know about P(x)?

P(x) = xR(x), if 2 is a room of P(x – 2), 0 is a root of P(x)

= x(x – 9)S(x), if 3 is a root of P(3x), P(9) = 0, so 9 is root of P(x)

= x(x – 9)(x – 4)T(x), if 4 is a root of 4P(x), 4 is a root of P(x)

$x(x-9)(x-4)(x-r)-1=(x-1)U(x)$, note $T(x)$ is of degree 1. Use 1st fact

Plug in 1:$1(1-9)(1-4)(1-r)-1=0$

$24(1-r)=1$

$1-r=1/24$

$R=23/24$

Answer is 23 + 24 = 47 choice D

2017 AMC 10A

Problem 24

For certain real numbers a, b, and c, the polynomial $g(x)=x^3+ax^2+x+10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x)=x^4+x^3+bx^2+100x+c$. What is $f(1)$?

$\textbf{(A)}\ -9009 \qquad \textbf{(B)}\ -8008 \qquad \textbf{(C)}\ -7007 \qquad \textbf{(D)}\ -6006 \qquad \textbf{(E)}\ -5005$

One key observation is that $f(x)=g(x)(x-r)$, because $f(x)$ has all roots of g, plus one more. Goal: find $f(1)$.

$f(x)=x^4+x^3+bx^2+100x+c=(x^3+ax^2+x+10)(x-r)$

$=x^4+ax^3+x^2+10x$

$-rx^3\ -arx^2-rx-10r$

$x^4+x^3+bx^2+100x+c=x^4+(a-r)x^3+(1-ar)x^2+(10-r)x-10r$

Equate coefficients of x, LHS = 100 , RHS = $10-r$

$100=10-r$

$r=-90$

Equate coefficients for x^3: $1 = a - r$

$1 = a - (-90)$

$1 = a + 90$

$A = -89$

$F(1) = g(1)(1 - r)$

$g(1) = 1^3 - 89(1)^2 + 1 + 10 = 1 - 89 + 11 = -77$

$f(1) = (-77)(1 - -90) = -77 \times 91$

f(1) = -7007, so the answer is choice C

2019 10 A

Problem 24

Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that $\dfrac{1}{s^3 - 22s^2 + 80s - 67} = \dfrac{A}{s-p} + \dfrac{B}{s-q} + \dfrac{C}{s-r}$ for all $s \notin \{p, q, r\}$. What is $\tfrac{1}{A} + \tfrac{1}{B} + \tfrac{1}{C}$?

(A) 243 **(B)** 244 **(C)** 245 **(D)** 246 **(E)** 247

What is another way to write

$x^3 - 22x^2 + 80x - 67 = (x - p)(x - q)(x - r)$

$$\frac{1}{(x-p)(x-q)(x-r)} = \frac{A}{(x-p)} + \frac{B}{(x-q)} + \frac{C}{(x-r)}$$

$$1 = A(x-q)(x-r) + B(x-p)(x-r) + C(x-p)(x-q)$$

What are good values/variables to plug into this equation?

Plug in $x = p$, $1 = A(p - q)(p - r) \rightarrow \frac{1}{A} = (p-q)(p-r)$

Plug in $x = q$, $1 = B(q - p)(q - r) \rightarrow \frac{1}{B} = (q-p)(q-r)$

Plug in x = r, 1 = C(r – p)(r – q) → $\frac{1}{C} = (r-p)(r-q)$

$$(p-q)(p-r)+(q-p)(q-r)+(r-p)(r-q)$$

$$p^2 - pr - qp + qr + q^2 - pq - qr + pr + r^2 - pr - rq + pq$$

$$p^2 + q^2 + r^2 - pr - pq - rq = 324 - 80 = 244$$

$x^3 - 22x^2 + 80x - 67 = (x - p)(x - q)(x - r)$

$(p + q + r)^2 = 22^2 = p^2 + q^2 + r^2 + 2pq + 2pr + 2rq$

$484 = p^2 + q^2 + r^2 + 2(80)$

$P^2 + q^2 + r^2 = 484 - 160 = 324$

So final answer is 244, Choice B

Number Theory

2018 10A

Problem 22

Let $a, b, c,$ and d be positive integers such that $\gcd(a, b) = 24$, $\gcd(b, c) = 36$, $\gcd(c, d) = 54$, and $70 < \gcd(d, a) < 100$. Which of the following must be a divisor of a?

(A) 5 **(B)** 7 **(C)** 11 **(D)** 13 **(E)** 17

What do I know about b? 24 | b, 36 | b … $2^3 3$, $2^2 3^2$, so $2^3 3^2 = 72$ | b

What do I know about c? 36 | c , 54 | b …$2^2 3^2$, $2(3^3)$, so $2^2 3^3 = 108$ | c

54 | d, 24 | a → 6 x 9, 8 x 3 in common 3 x 2 = 6 (what we know for sure) gcd(d, a) = 72, 78, 84, 90 and 96, divide by 6

12 13 14 15 16

A can't have another factor of 3 → 12 and 15 are out

D can't have another factor of 2 → 12, 14 and 16 are out

Can we rule out anything? → E can be ruled out 17 doesn't divide into our possibilities.

CHOICE D = 13

Probability

2019 10 A

Problem 20

The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

$\textbf{(A) }\frac{1}{21}\qquad\textbf{(B) }\frac{1}{14}\qquad\textbf{(C) }\frac{5}{63}\qquad\textbf{(D) }\frac{2}{21}\qquad\textbf{(E) }\frac{1}{7}$

5 ODD, 4 EVEN

Each row must have how many odd numbers? 1 or 3

O O O

O E E

O E E Valid….

We are selecting 5 slots out of 9 for an odd number Sample space is 9 choose 5 or 9 choose 4.

Row 2, col 3

E E O

O O O

E E O

How many possible valid arrangements? 3 x 3

Answer = 9/(9 choose 5) = 9/126 = 1/14

CHOICE B

Problem 22

Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

$\textbf{(A) } \frac{1}{3} \qquad \textbf{(B) } \frac{7}{16} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } \frac{9}{16} \qquad \textbf{(E) } \frac{2}{3}$

Scenario 1 → 50%, Scenario 2 → 50%

We play twice:

Scenario 1 both times (flip second) → 1/4

Each scenario once → 1/2

Scenario 2 both times (rnd number in range) → ¼

¼ of the time we pick either 0 or 1. When is the difference between our two choices going to be more than ½?

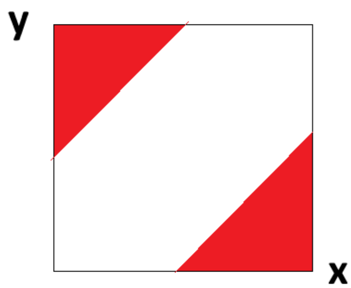
The difference will be either 0 (with probability ½) or 1 (with probability ½)

Total probability here is ¼ x ½ = 1/8

Each scenario once: 1 number is either 0 or 1, the other number is randomly selected from 0 to 1. Probability difference is more than ½ is ½ because 0 and 1 are at the end points, we have a 50% of being on the other half.

Total probability here = ½ x ½ = ¼

Scenario 2 twice, 2 randomly selected numbers from 0 to 1. See picures:



The difference between x and y is greater than 1/2 in the shaded red areas. Each pt in the square is equally probable. So our probability of getting a difference greater than 1/2 is the red area divided by the total area.
Red area = 1/4

Probability here is ¼ x ¼ = 1/16

Total answer = 1/8 + ¼ + 1/16 = 7/16

This is CHOICE B.