<u>OMC Combinatorics Homework Solutions (for 9/27/2024)</u>

1. Use a combinatorial argument to prove that $\binom{n}{k} = \binom{n}{n-k}$.

<u>Solution</u>
The left hand side represents the number of ways to choose k items out of n. Every time one chooses k items, there are n – k items they **<u>didn't select</u>**. Thus, there's a one to one correspondence between any combination of k items out of n to a combination of n – k items out of n. For example, consider the example of n = 5 with the items being A, B, C, D, E with k = 3 and the particular combination {A, C, D}. This corresponds to the subset {B, E} of the 5 – 3 = 2 items that weren't selected in the subset of 3 listed originally. Any time you can show a one to one correspondence between two finite sets, that means that the sets are the same size.

2. Use a combinatorial argument to prove that $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$. (Hint: consider counting the number of ways to choose n objects out of 2n objects, where n of the objects are painted red and the other n objects are painted blue. Break up your counting into n+1 different bins depending on the number of objects in your subset of size n that are painted red.)

<u>Solution</u>
Let there be 2n items from which we are selecting n. There are $\binom{2n}{n}$ ways to do this, by definition. Now, imagine painting n of the items red and the other n items blue. A different way to count the number of subsets of n items out of the 2n total items is to break them into categories:

Category 0: 0 red items, n blue items
Category 1: 1 red item, n – 1 blue items
Category 2: 2 red items, n – 2 blue items
...
Category i: i red items, n – i blue items
...
Category n: n red items, 0 blue items

We can select the subsets in category i in $\binom{n}{i} \times \binom{n}{n-i}$, first choosing the i red items, then choosing the n – i blue items and then putting together each set of red items with each set of blue items. But, using the result from question #1, we see that

$$\binom{n}{i} \times \binom{n}{n-i} = \binom{n}{i} \times \binom{n}{i} = \binom{n}{i}^2$$

Now, to get all possible subsets of size n, just sum over all possible values of i. Thus, the number of ways to choose n items out of 2n is also equal to

$$\sum_{i=0}^{n} \binom{n}{i}^2$$

This proves the given identity.

3. A yogurt shop has 4 different yogurt flavors and 6 different toppings. If a customer can choose a desert with exactly 1 yogurt flavor and 2 different toppings, how many different desserts can the customer choose from?

Solution

We can combine any choice of yogurt flavors with any choice of toppings so we have: $\binom{4}{1} \times \binom{6}{2} = 4 \times 15 = 60$

4. In a Math Counts countdown round, the 10th seeded mathlete goes against the 9th seeded mathlete, the loser of the showdown gets awarded 10th place and the winner goes against the 8th seeded mathlete. The loser of that showdown gets awarded 9th place and the winner goes onto face the 7th seeded mathlete. The showdowns continue in this fashion until the last one, where the previous winner goes against the 1st seeded mathlete. The winner of that showdown wins the tournament and the loser gets 2nd place. In how many different orders could the 10 seeded mathletes finish the competition?

Solution

There are 9 matches, and each match has 2 possible outcomes (win or loss for the challenger). Every one of the 2^9 possible outcomes of these 9 matches results in a different ordering of the participants. Thus there are 512 possible outcomes of the competition.

5. What is the sum of all four digit palindromes? A palindrome is a number that reads the same forwards and backwards. For example, 2332 should be one of the palindromes that is part of the sum for this question.

Solution

The first digit can be anything in between 1 and 9, inclusive.
The second digit can be anything in between 0 and 9, inclusive.
Thus, there are 9 x 10 = 90 choices of values for the first two digits.
The third digit must equal the second digit, so there's 1 choice for it.
the last digit must equal the first digit, so there's 1 choice for it.

Thus, we want the sum of the 90 four digit palindromes. Each digit from 1 to 9 appears in the thousands digit 10 times and the ones digit 10 times. Summing up these contributions we get $10 \times 1001 \times \left(\sum_{i=1}^{9} i\right) = 10010 \times 45 = 450450$.

Each of the digits 0 through 9 appear in both the hundreds and tens place 9 times. Summing up these contributions we get

$$9 \times 110 \times \left(\sum_{i=1}^{9} i\right) = 990 \times 45 = 45000 - 450 = 44550$$

Adding this all up, we get **495,000.** More easily, if we factor out 45 from both expressions, we get $45 \times (10010 + 990) = 45 \times 11000 = 495000$.

6. A new school has exactly 1000 students and 1000 lockers. Number the lockers 1, 2, 3, …, 1000. The lockers are all initially closed. On the first day, the first student opens all the lockers. On the second day the second student closes every other locker, starting at 2. On the third day, the third student changes the state (open to close or close to open) of every third locker starting at locker 3. This continues (the ith student changes the state of one out of every i lockers starting with locker number i). At the end of this process, which lockers will be open?

Solution

The state of locker i is changed on each day that is a divisor of i. If i has an even number of divisors, at the end of the whole process, locker i will be closed. If i has an odd number of divisors, at the end of the process, locker i will be open. The only numbers with an odd number of divisors are perfect squares, so the only lockers open at the end of the process will be the perfect squares from 1^2 through $31^2 = 961$, inclusive.

7. Evaluate the following sum: $\sum_{i=1}^{n} i\binom{n}{i}$.

Solution

$$\sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=1}^{n} \frac{i \times n!}{i! \times (n-i)!}$$

$$= \sum_{i=1}^{n} \frac{n!}{(i-1)! \times (n-i)!}$$

$$= n\sum_{i=1}^{n} \frac{(n-1)!}{(i-1)! \times (n-i)!}$$

$$= n\sum_{i=1}^{n} \binom{n-1}{i-1}$$

$$= n\sum_{i=0}^{n-1} \binom{n-1}{i}$$

$$= n2^{n-1}$$

8. Evaluate the following sum: $\sum_{i=0}^{45}(44-i)\binom{45}{i}$.

Solution

$$\sum_{i=0}^{45}(44-i)\binom{45}{i} = \left[\sum_{i=0}^{45} 44\binom{45}{i}\right] - \left[\sum_{i=0}^{45} i\binom{45}{i}\right]$$

$$= \left[44\sum_{i=0}^{45}\binom{45}{i}\right] - 45(2^{44})$$

$$= [44(2^{45})] - 45(2^{44})$$

$$= [44(2)(2^{44})] - 45(2^{44})$$

$$= 88(2^{44}) - 45(2^{44})$$

$$= (88 - 45)(2^{44})$$

$$= 43 \times 2^{44}$$

9. (2014 AMC 12A Q 13) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

Solution

We can assign each person to any one of five rooms, so this is a total of 5^5 assignments. (This is because both the rooms and the friends are distinguishable.) But, some of these assignments aren't allowed. In particular, any assignment with 3 people in one room, or four people in one room or five people in one room aren't allowed. So let's subtract those out.

5 people in one room = 5 ways (all in room 1 or all in room 2, etc.)

4 people in room A, 1 person in room B = there are $\binom{5}{4}$ ways to choose the people for the crowded room, 5 choices for the crowded room, and then 4 choices for the solo room. This is a total 5 x 5 x 4 = 100 ways the people can be in the rooms with 4 people in one room.

3 people in room A, 1 person in room B, 1 person in room C = there are $\binom{5}{3}$ to ways to choose the people for the crowded room, 5 choices for the crowded room, 4 choices for the room for the first solo person and then 3 choices for the room for the second solo person. This is a total of 10 x 5 x 4 x 3 = 600 ways this could happen.

3 people in room A, 2 people in room B = there are $\binom{5}{3}$ to ways to choose the people for the crowded room, 5 choices for the crowded room, and 4 choices for the less crowded room. This is a total of 10 x 5 x 4 = 200 ways this could occur.

Our final answer is $5^5 – 5 – 100 – 600 – 200 = 3125 – 905 = 2220$.

10. A permutation of the integers 1, 2, 3, …, n is called a mountain permutation if the first k items of the permutation are all in increasing order, with the kth item equal to n (the peak and the remaining items are all in decreasing order. An example of a mountain permutation for n = 8 is 2, 3, 4, 7, 8, 6, 5, 1. Note that a mountain permutation may either begin or end with n (so 1, 2, 3, 4, 5, 6, 7, 8 is a valid mountain permutation.) How many mountain permutations are there of 1, 2, 3, …, n, in terms of n?

Solution

The value n must be the "peak" of the permutation. For all other values, 1 through n-1, the value can either be to the left of the peak or the right of the peak. In short, there are 2 choice for where to place it. Once we make the left/right choice for each value from 1 to n – 1, the permutation is forced due to the mountain permutation ordering. Thus, there must be exactly 2^{n-1} mountain permutations. As an example, consider n = 5 and the choices 1 = L, 2 = L, 3 = R and 4 = L. These 4 choices are i n one to one correspondence with the mountain permutation 1, 2, 4, 5, 3.