OMC Probability Hwk Solutions Due 11/3/23

2010 12A Q15) $\binom{4}{2} p^2(1-p)^2 = \frac{1}{6}$, via Bin Dist p = heads

$\boxed{D}$

$$\sqrt{p^2(1-p)^2} = \sqrt{\frac{1}{36}}$$

$$p(1-p) = \pm\frac{1}{6}$$

$$p - p^2 = \frac{1}{6} \quad \text{or} \quad p - p^2 = -\frac{1}{6}$$

$$p^2 - p + \frac{1}{6} = 0$$

$$6p^2 - 6p + 1 = 0$$

$$p = \frac{6 \pm \sqrt{36 - 4(6)(1)}}{2 \times 6} = \frac{1}{2} \pm \frac{2\sqrt{3}}{12} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$$

Since $p < \frac{1}{2}$, $p = \frac{1}{2} - \frac{\sqrt{3}}{6} = \boxed{\frac{3-\sqrt{3}}{6}}$ $\boxed{(D)}$

2010 12A Q)16) Bernardo sample space $= \binom{9}{3} = 84$

$\boxed{B}$ He chooses 9 in $\binom{8}{2} = 28$ combos.

So w/probability $\frac{28}{84} = \frac{1}{3}$ he automatically wins

Now, let's analyze what occurs $\frac{2}{3}$ of the time: a Symmetric game. Probability of a tie is

$\frac{1}{\binom{8}{3}} = \frac{1}{56}$, because each of 56 combos is equally likely.

Prob someone wins $= 1 - \frac{1}{56} = \frac{55}{56}$.

Prob Bernardo wins $= \frac{1}{2} \times \frac{55}{56} = \frac{55}{112}$ (symmetric)

Total Prob Bernardo $= \frac{1}{3} + \frac{2}{3} \times \frac{55}{112} = \frac{112 + 110}{3 \times 112}$

$$\frac{222}{3 \times 112} = \frac{3 \times 2 \times 37}{3 \times 2 \times 56} = \boxed{\frac{37}{56}} \text{ B}$$

$\boxed{E}$ 2010 12B Q11) This asks about all 4 digit palindromes, which are of the form abba w/ $1 \le a \le 9$, $0 \le b \le 9$.
$= 1001a + 110b$, since $7 \mid 1001$, abba is divisible by 7 iff $7 \mid 110b$. But $7 \nmid 110$ and 7 is prime so this only occurs when $7 \mid b$. Given the range of b, there are 2 solutions: $b=0, b=7$, out of 10 options. The desired probability is $\boxed{\frac{1}{5}}$ (E)

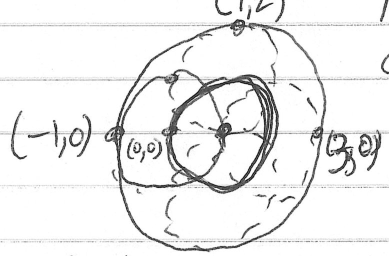
$\boxed{E}$ 2010 12B Q16) $abc + ab + a = a(bc + b + 1)$.
Thus we want probability $3 \mid a$ or $3 \mid (bc+b+1)$.

$3 \mid a$ w/probability $\frac{1}{3}$. Since all of these cases are covered, we want to add $\frac{2}{3} \times p(3 \mid (bc+b+1))$.
if $b \equiv 0 \pmod 3$ $bc+b+1 \equiv 0+0+1 \equiv 1 \mod 3$
if $b \equiv 1 \pmod 3$ $bc+b+1 \equiv c+2 \mod 3$, which is 0 iff $c \equiv 1 \pmod 3$
if $b \equiv 2 \pmod 3$ $bc+b+1 \equiv 2c+3 \equiv 2c \mod 3$, which is 0 iff $c \equiv 0 \pmod 3$
Thus $p(3 \mid (bc+b+1)) = \frac{2}{9}$

Final Answer $= \frac{1}{3} + \frac{2}{3} \times \frac{2}{9} = \frac{9+4}{27} = \boxed{\frac{13}{27}}$ $\boxed{E}$

C

2010 12B Q18) Let the frog start at (0,0). Without loss of generality, let the frog land at (1,0) after its first jump. Now after its second jump the frog will land on the circle $(x-1)^2+y^2=1$:
(Shaded in)



Now, imagine drawing a circle ~~for each~~ w/radius 1 for each point on $(x-1)^2+y^2=1$ as the center of the circle.

The result of the union of these circles is a larger circle of radius 2 with center (1,0). The frog could land anywhere is this circle with equal probability (due to symmetry). Of these places the original circle has area π. Thus, the desired probability is $\frac{\pi}{4\pi} = \boxed{\frac{1}{4}}$ $\boxed{C}$

B

2012 12A Q11) Alex $=\frac{1}{2}$, Mel $=\frac{1}{3}$, Chelsea $=\frac{1}{6}$, since $\frac{1}{3}=2\times\frac{1}{6}$ and $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$. Desired probability is

$$\binom{6}{3,2,1}\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right)^1 = \frac{6!}{3!2!}\times\frac{1}{8}\times\frac{1}{9}\times\frac{1}{6}$$

$$\boxed{B} = \frac{120}{6\times2\times72} = \frac{20}{144} = \boxed{\frac{5}{36}}$$

A

2012 12A Q15)

1	2	3
8	B	4
7	6	5

$\rightarrow$

7	8	1
6	B	2
5	4	3

$1\to3$ $3\to5$ $5\to7$ $7\to1$
$2\to4$ $4\to6$ $6\to8$ $8\to2$

Thus $p(1 \text{ or } 3)=B \wedge p(3 \text{ or } 5)=B \wedge p(5 \text{ or } 7)=B \wedge p(7 \text{ or } 1)=B$ is probability all "odd" squares end black. Sample space $=2^4=16$ of these 7 work (1,5), (3,7), (1,3,5), (1,3,7), (1,5,7), (3,5,7) and (1,3,5,7).

2012 12A Q15 cont) It follows that probability 2,4,6,8 ALSO end up black is $\frac{7}{16}$.
Finally middle is black with probability $\frac{1}{2}$. $P(\text{all black}) = \frac{1}{2} \times \frac{7}{16} \times \frac{7}{16} = \boxed{\frac{49}{512}}$ $\boxed{A}$

2013 12A Q22) Let's work this backwards. Start with ABCBA, multiply by 11 to get

$\boxed{E}$

$$\begin{array}{r} ABCBA0 \\ +\ ABCBA \\ \hline A(A+B)(B+C)(C+B)(B+A)\,A \end{array}$$

$A+B<10$ because the carry messes up units match

Similarly, $B+C<10$
Given $A \geq 1$ Count # solutions
if $A=1$, $B=0 \rightarrow C=(0,9)$, $B=1 \rightarrow C=(0,8)$, etc. last sol is $B=8$
This is $10+9+\cdots+2 = 54$ sols.
if $A=2$, $B=0 \rightarrow 7 \Rightarrow 10+9+\cdots 3 = 52$ sols
Pattern is

$$\begin{array}{l} 10+9+\cdots+2 \\ +\ 10+9+\cdots\ 3 \\ +\ 10+9\cdots 4 \\ +\ \vdots \\ \quad 10 \\ \hline 9(10)+8(9)+7(8)+\cdots 1(2) \end{array}$$

$$\sum_{i=1}^{9} i(i+1) = \frac{9\times10\times11}{3} = \boxed{330}$$

Sample space is all palindromes of length 6 ABCCBA, which is 9 choices for A, 10 for B and C, $9\times10\times10=900$

Final answer $= \frac{330}{900} = \frac{33}{90} = \frac{11}{30}$ $\boxed{E}$

$\boxed{C}$

2014 12B (Q22) Let $p(k)$ be the probability of winning from position k. Game is symmetric about $k=5$ so $p(5)=\frac{1}{2}$. Here are the equations:

$$\begin{aligned}
&(a) \quad p(1) = \tfrac{9}{10}\,p(2) \qquad \text{(can only win going right)}\\
&(b) \quad p(2) = \tfrac{2}{10}\,p(1) + \tfrac{8}{10}\,p(3)\\
&(c) \quad p(3) = \tfrac{3}{10}\,p(2) + \tfrac{7}{10}\,p(4)\\
&(d) \quad p(4) = \tfrac{4}{10}\,p(3) + \tfrac{6}{10}\times\tfrac{1}{2}
\end{aligned}$$

$$p(2) = \frac{2}{10}\left(\frac{9}{10}p(2)\right) + \frac{8}{10}\,p(3) \quad \text{[sub (a) in (b)]}$$

$$p(2) = \frac{9}{50}p(2) + \frac{4}{5}p(3)$$

$$\frac{41}{50}p(2) = \frac{4}{5}p(3) \longrightarrow \boxed{p(2) = \frac{40}{41}p(3)}$$

$$p(3) = \frac{3}{\cancel{10}} \times \frac{\overset{4}{\cancel{40}}}{41}p(3) + \frac{7}{10}p(4)$$

$$\frac{29}{41}p(3) = \frac{7}{10}p(4) \longrightarrow \boxed{p(3) = \frac{7\times 41}{29\times 10}p(4)}$$

$$p(4) = \frac{4}{10}\left(\frac{7\times 41}{29\times 10}p(4)\right) + \frac{3}{10}$$

$$p(4) = \frac{287}{725}p(4) + \frac{3}{10}$$

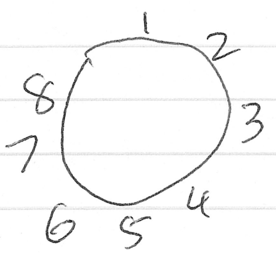
$$\frac{498}{725}p(4) = \frac{3}{10} \longrightarrow p(4) = \frac{3}{10}\times\frac{725}{498}$$

Thus, $$p(1) = \frac{9}{\cancel{10}} \times \frac{\overset{4}{\cancel{40}}}{\cancel{41}} \times \frac{7\times\cancel{41}}{\cancel{29}\times 10} \times \frac{3}{\underset{2}{\cancel{10}}} \times \frac{\overset{5}{\cancel{725}}}{498} \quad \cancel{145}$$

$$p(1) = \frac{9\times\cancel{4}\times 7\times\cancel{3}\times\cancel{5}}{\underset{2}{\cancel{10}}\times\cancel{2}\times\underset{166}{\cancel{498}}} = \boxed{\frac{63}{166}} \quad \boxed{C}$$

$\boxed{A}$

2015 12A Q17) Sample Space $= 2^8 = 256$



0H = 1 way yes

1H = 8 ways yes

2H = $\binom{8}{2} - 8 = 20$

because we can't do (1,2), (2,3), ... (8,1).

4H = 2 ways (1357 or 2468)

3 is harder. Do case work.

(1,3,5) (1,3,6), (1,3,7), (1,4,6), (1,4,7), (1,5,7) all with 1

(2,4,6) (2,4,7), (2,4,8), (2,5,7), (2,5,8), (2,6,8) all with 2

(3,5,7), (3,5,8), (3,6,8) all w/3 without 1,2

(4,6,8) all w/4 not 1,2,3

total = 6+6+3+1 = 16

total = 1+8+20+16+2 = 47 ways

prob = $\frac{47}{256}$ $\boxed{A}$

$\boxed{D}$

2015 12B Q17) Using binomial distribution we have

$$\binom{n}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2} = \binom{n}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$$

$$\frac{\cancel{n(n-1)}}{2} \times \frac{3}{\cancel{4}} = \frac{\cancel{n(n-1)}(n-2)}{6} \times \frac{1}{\cancel{4}}$$

$$9 = n-2$$

$$\boxed{11 = n} \quad \boxed{D}$$

B

2016 12A (Q19) Sample Space $= 2^8$

8H $\to$ 1 ways ✓ yes

7H $\to$ 8 ways ✓ yes

6H $\to$ $\binom{8}{2}$ ways $= 28$ yes since end at $x=4$

4H $\to$ 1 way HHHH TTTT

Let's see how many ways to get to $x=4$ with 5H:

$\text{HHHH}\boxed{} \to \binom{4}{3} = 4$ ways

$\boxed{3H\,1T}\text{HHTT} \to \binom{4}{1} = 4$ ways

$\Big\}$ 8 ways

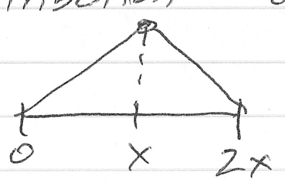
If you get 2H 2T in first 4, can't be done.

Total $= 1 + 8 + 28 + 8 + 1 = 46$

prob $= \frac{\cancel{48}\,46}{256} = \frac{23}{128}$, $a+b = 151$ (B)

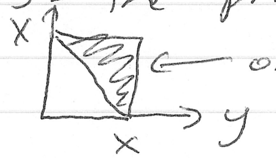
C

2016 12A (Q23) There are methods using Calculus to solve this but there's one easier way. Let x be the largest side. Then, both y, z are uniformly distributed in the range $[0, x]$. The probability $y+z > x$ is $\frac{1}{2}$ since the distribution of $y+z$ will look like this



symmetric

Thus the probability is $\frac{1}{2}$.



okay

$\boxed{C}$

B

2016 12B Q19) Sum over game ending on turn 1, 2, 3, ...

prob end in success in k turns $= \left(\frac{1}{2}\right)^{3k}$, all 3 must flip $(k-1)$ tails followed by 1 H.

Desired probability $=$

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{3k} = \sum_{k=1}^{\infty} \left(\frac{1}{8}\right)^{k} = \frac{\frac{1}{8}}{1-\frac{1}{8}} = \boxed{\frac{1}{7}}\ \boxed{B}$$