Example Probability Problems Worked Out

Problem (2021 Fall 12A Q 18)

Each of the 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 4 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 16$

Solution

Imagine that we write down which bin each ball goes to (label bins A, B, C, D , E):

AAABBBBBCCCCDDDDEEEE (counting the # of permutations of this)

$$\frac{20!}{3!\,5!\,4!\,4!\,4!}$$

But this forces 3 balls in bin A and 5 in bin B, we could just as easily have done 3 in bin B and 5 in bin A, etc. How many different ways can we assign the bin that gets 3 balls and 5 balls? Total = 5 x 4 = 20 choices…so that means the total number of permutations with that distribution is

$$\frac{20! \times 20}{3!\,5!\,4!\,4!\,4!}$$

# of ways to get 4 in every bin

$$\frac{20!}{4!\,4!\,4!\,4!\,4!}$$

Do the division, the 20! cancel as do 3 of the 4! … $\frac{\frac{20}{3!5!}}{\frac{1}{4!4!}} = \frac{20 \times 4! \times 4!}{3! \times 5!} = \frac{80}{5} = 16$

Answer $=$ E

Problem 11 (2021 Fall 12B)

Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

$\textbf{(A)}\ \frac{3}{4} \qquad \textbf{(B)}\ \frac{57}{64} \qquad \textbf{(C)}\ \frac{59}{64} \qquad \textbf{(D)}\ \frac{187}{192} \qquad \textbf{(E)}\ \frac{63}{64}$

Solution

Probabilty ALL ODD OR Probability exactly one value of 2 or 6.

P(ODD) = $\frac{1}{64}$

P(5 ODD, 1 die 2 or 6) = $\binom{6}{1}(\frac{1}{2})^5\frac{1}{3} = \frac{6}{32\times3} = \frac{1}{16} = \frac{4}{64}$

Subtract and get choice **C**

Problem 11 (2020 12A)

A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0,0), (0,4), (4,4),$ and $(4,0)$. What is the probability that the sequence of jumps ends on a vertical side of the square$?$

$\textbf{(A)}\ \frac{1}{2} \qquad \textbf{(B)}\ \frac{5}{8} \qquad \textbf{(C)}\ \frac{2}{3} \qquad \textbf{(D)}\ \frac{3}{4} \qquad \textbf{(E)}\ \frac{7}{8}$

Solution

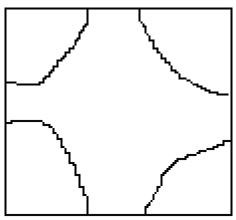
	¼ gets here			
¼ gets here	**frog**	**1/4 gets here**		
	¼ gets here			

Probability vertical = ¼ + 1/4*1/2 + ½*1/2 = ¼ + 1/8 + ¼ = **5/8 choice B**

Problem 16 (2020 12A)

A point is chosen at random within the square in the coordinate plane whose vertices are $(0,0), (2020,0), (2020,2020),$ and $(0,2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x,y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

$\textbf{(A)}\ 0.3 \qquad \textbf{(B)}\ 0.4 \qquad \textbf{(C)}\ 0.5 \qquad \textbf{(D)}\ 0.6 \qquad \textbf{(E)}\ 0.7$



Area of the four quarter circles that are within distance d from a Lattice point is $\pi d^2 = \frac{1}{2}$

$d^2 = \frac{1}{2\pi} \sim \frac{1}{6.28} \sim \frac{1}{6.25} \rightarrow d \sim \frac{1}{2.5} = 0.4$ (choice B)

Problem 17 (2021 Fall 12B)

A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?

$\textbf{(A)}\ \frac{13}{108} \qquad \textbf{(B)}\ \frac{7}{54} \qquad \textbf{(C)}\ \frac{29}{216} \qquad \textbf{(D)}\ \frac{4}{27} \qquad \textbf{(E)}\ \frac{1}{16}$

Solution

In 2 steps, with probability 1/6 we are back at the beginning. Probability 2/6 we are one away from where we started.

0→1→1→1→1→(0,1)   1/3 * 1/3*1/3*1/2 = 1/54

0→1→1→1→0→1   1/3* 1/3*1/6*1 = 1/54

0→1→1→0→1→(0,1)   1/3*1/6*1*1/2 = 1/36

0→1→0→1→1→(0,1)   1/6*1*1/3*1/2 = 1/36

0→1→0→1→0→1 1/6*1*1/6*1 = 1/36

4/108 + 9/108 = 13/108

Problem 15 (2013 12A)

Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

$\textbf{(A)}\ 96 \qquad \textbf{(B)}\ 108 \qquad \textbf{(C)}\ 156 \qquad \textbf{(D)}\ 204 \qquad \textbf{(E)}\ 372$

If both Peter and Pauline go to the same store (this can happen in 4 ways), then all 3 kids can be distributed to any of 3 stores in 3^3 ways.= 4 x 27 = 108

If Peter and Pauline go to different stores, we have 4 choices for Peter, 3 choices for Pauline, and then 2 choices for each of the kids = 4 x 3 x 2 x 2 x 2 = 96

108 + 96 = 204

Problem 15 (2013 12B)

The number 2013 is expressed in the form

$$2013 = \frac{a_1!a_2!...a_m!}{b_1!b_2!...b_n!},$$

where $a_1 \ge a_2 \ge ... \ge a_m$ and $b_1 \ge b_2 \ge ... \ge b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 2 \qquad \textbf{(C)}\ 3 \qquad \textbf{(D)}\ 4 \qquad \textbf{(E)}\ 5$

Solution

2013 = 3 x 671 = 3 x 11 x 61

If we want a1 and b1 to be small, we know that we need at least 61! Somewhere because there's no other way to get a copy of 61. We don't need anything bigger…

$$\frac{61!}{60!} \times \frac{11!}{10!} \times \frac{3!}{2!} \times \frac{5!}{5!}$$

If we want to do better, we must cancel 60 from 60! To make it 59!

$$\frac{61!}{60!} \times \frac{11!}{10!} \times \frac{3!}{2!} \times \frac{10}{10} = \frac{61!}{59! \times 60} \times \frac{11!}{10!} \times \frac{60}{20} = \frac{61!}{59!} \times \frac{11!}{10!\,20} \times \frac{6}{6} = \frac{61!\,11!\,3!}{59!\,10!\,5!}$$

A1 = 61, b1 = 59, so the answer is 61 – 59 = 2.