Some Probability Reminders/Basics

Expected Value – how many turns on average does a game last?

Alice and Bob are shooting at a target. Alice hits it with probability 1/2

Bob hits it with probably ¼. They alternate turns until one person hits the target, with Alice starting first.

(a) What is Alice's probability of winning

(b) What is the expected # of turns in the game (total shots taken)

Let X = probability that Alice wins

X = ½ + (1 / 2) * (3/ 4)*X

Either Alice wins outright, OR she misses and Bob misses and we return the game to the beginning

X = ½ + (3/8)*X

(5/8)*X = ½

X = (1 / 2) / (5/ 8) = **4/5.**

Expectation = p(X = 1)*1 + p(X = 2)*2 + p(X = 3)*3…

Let Y = expected number of turns of the game.

Y = (1/ 2)* 1 + (1/ 2)*(1/4)*2 + (1/2)*(3/4)*(Y + 2)

Y = ½ + ¼ + (3/8)*(Y + 2)

Y – 3/8*Y = 1 / 2 + 1/ 4 + 3/ 4

5/8 * Y = 3/2 → Y = (3/2) / (5/8) = 1.6

S = 1 x (1 / 2) + 2 x (1/ 4) + 3 x (1/ 8) + 4 x (1/ 16)

S/2 = 1 x (1/4) + 2 x (1/8) + 3 x (1/16)

S – S/2 = 1 / 2 + 1/ 4 + 1/ 8 + 1/ 16

S/2 = 1

S = 2

Question: Let X be a randomly chosen divisor of 10^{99}, what is the probability that X is a multiple of 10^{88}?

# of divisors of $10^{99} = 2^{99} \times 5^{99}$ → 100 x 100 = 10,000

How many of these are multiples of 10^{88}? $2^{88+a}5^{88+b}$, how many different values for a and b? a 0 .. 11 = 12 values, b = 0 .. 11 = 12 values → 12 x 12

Final answer = $\frac{12\times12}{100\times100} = \frac{9}{625}$

Next item: biased coins and the binomial distribution

The probability of getting heads is p. We flip a coin 7 times, the probability of getting exactly 2 heads equals the probability of getting exactly 3 heads. What is p?

Binomial distribution is repeating a yes/no process n times, where each trial has an independent probability of p of success.

Probability we get exactly k successes is $\binom{n}{k} p^k (1-p)^{n-k}$

$$\binom{7}{2} p^2 (1-p)^5 = \binom{7}{3} p^3 (1-p)^4$$

$$21(1-p) = 35p$$

$$21 - 21p = 35p$$

$$21 = 56p$$

$$p = 3/8$$

Lily Pad Problems

frog		**Has to gethere**	**eaten**	**14/27**			**home**
__ 0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

When the frog jumps with probability 1/3 he jumps 1 square, probability 2/3 he jumps 2 squares. What is the probability he reaches home EXACTLY.

P(1) = 1/3 (on one jump)

P(2) = 2/3 + 1/3 * 1/3 (either 1 or 2 jumps) = 7/9

P(4) = (7/9) * (2/3) = 14/27

Now, from this point on what is his probability of going exactly 3 squares.

P(3) = p(1)*p(1)*p(1) + p(1)*p(2) + p(2)*p(1)

P(3) = p(1)*p(1)*p(1) + 2*p(1)*p(2)

= 1/3 * 1/3 *1/3 + 2* (1/3) * (2/3) = 1/27 + 4/9 = 13/27

P(home) = (14/27)*(13/27)

New frog problem

		Frog		
0	1	2	3	4

On each turn probability ½ goes 1 step left, ½ goes 1 step right.

What's the probability the frog ends up at square 4 in exactly 4 moves.

Probability distribution of where the frog will be in 2 steps starting at square 2.

Probability distribution of where the frog will be going "backwards" in 2 steps starting at square 4.

In 2 steps. ½ probability back at square 2, ¼ probability at square 0, ¼ probability at square 4

In 2 steps $\frac{1}{2}$ probability back at 4, $\frac{1}{4}$ probability at 2

Probability at 4 in 4 steps = p(get to 2 in 2)*p(get from 2 to 4 in 2 steps) = $\frac{1}{2}$*1/4

+ p(get to 4 in 2)*p(get from 4 to 4 in 2 steps) = $\frac{1}{4}$*1/2