Mathcounts National Competition Practice Sols
Sprint Round

1) $\boxed{4}$

$$10a + b + 3a + 3b = 10b + a$$ $\boxed{b = 1, 2, 3, 4}$

$$13a + 4b = 10b + a$$
$$12a = 6b$$
$$2a = b$$

2) $\boxed{65^\circ}$

$\measuredangle C \geq 58^\circ$ $\measuredangle A + \measuredangle C \geq 115^\circ \Rightarrow \measuredangle B \leq 65^\circ$

3) $\boxed{2017}$

$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$, thus $n = 2017$, telescopic sum

$\downarrow$

$\frac{1}{i} - \frac{1}{i+1}$, so $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \cdots + \frac{1}{2017} - \frac{1}{2018} = \frac{2017}{2018}$

4) $\boxed{25}$

$$m^2 - 100 = 19m + 50$$
$$m^2 - 19m - 150 = 0$$
$$(m-25)(m+6) = 0 \rightarrow m = 25$$

5) $\boxed{5}$

$$27x^3 - 3x^3 = 3000$$
$$24x^3 = 3000$$
$$x^3 = \frac{3 \times 2^3 \times 5^3}{3 \times 2^3} \rightarrow x = 5$$

6) $\boxed{60}$

2 cats catch 3 mice in 5 days $\rightarrow$ 1 cat catches 3 mice in 10 days

20 cats can catch 60 mice in 10 days

Do $\boxed{20 \times 3}$

7) $\boxed{\sqrt{58}}$

$(10,4) \to (17,7)$ $\sqrt{3^2+7^2} = \sqrt{58}$

Just do gcd(3,7)...

8) $\boxed{75}$

$45+3x = .8(50+5x)$

$45+3x = 40+4x$

$5 = x$

5x = # T/F

T/F = 25

45 of 50

15 of 25

60 of 75 ✓

9) $\boxed{10}$

$5^{2017}+5 = 5(5^{2016}+1) \equiv 5(1+1) \mod 12$

$= \boxed{10}$

$5^2 \equiv 1 \pmod{12}$

alteratively $5^0, 5^1, 5^2$, etc alternates 1, 5, 1, 5, etc mod 12

so $5^{2017} \equiv 5 \pmod{12}$

10) $\boxed{495}$

$$\begin{array}{r} 942 \\ -249 \\ \hline 693 \end{array} \quad \begin{array}{r} 963 \\ -369 \\ \hline 594 \end{array} \quad \begin{array}{r} 954 \\ -459 \\ \hline \boxed{495} \end{array}$$

11) $\boxed{11}$

x couples x men x women

$\underset{\text{men}}{x}(\underset{\text{women}}{x-1}) + \binom{x}{2} = x(x-1) + \frac{x(x-1)}{2} = 165$

2 men

$\frac{3}{2}(x)(x-1) = 165$

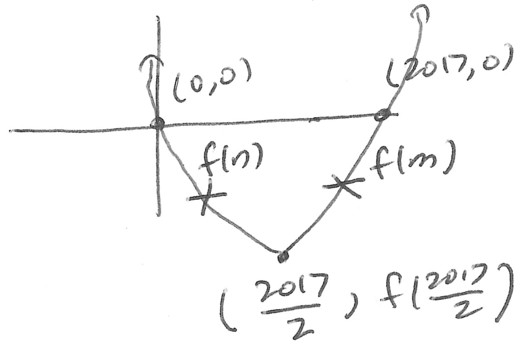
$x(x-1) = \frac{15 \times 11 \times 2}{3}$

$= 10 \times 11$

$\boxed{x = 11}$

12) $\boxed{94}$ $(1,7),(2,8),\ldots(94,100)$

13) $\boxed{0}$



$m = \frac{2017}{2} + x$

$n = \frac{2017}{2} - x$

$f(n+m) = f(2017) = 0$

~~14) 20~~ ~~11, 4, 16, 37, 58, 89, 145, 42, 20, 4~~

~~1 2 3 4 5 6 7 8 9 10~~

~~period = 9 8~~

~~4 is term 2, 10, 18, etc 2 mod 8~~

~~$8\overline{)2016}$ 252 ✓ 2017 ≡ 1 mod 8~~

41

14) $\boxed{169}$ 11, 4, 16, 49, 169, 256, 169

1 2 3 4 5 6 7

odd = 169, even = 256

15) $\boxed{\frac{195}{8}}$

$$\begin{aligned} a_1+a_2+a_3+a_4 &= 12 \\ a_5+a_6+a_7+a_8 &= 9 \\ -\ a_9+a_{10}+a_{11}+a_{12} &= 6 \\ \hline -8d-8d-8d-8d &= 6 \\ -32d &= 6 \\ d &= -\frac{3}{16} \end{aligned}$$

$a_9+a_9+a_9+a_9-\frac{18}{16} = 6$

$4a_9 = 6 + \frac{9}{8}$

$a_9 = \frac{3}{2} + \frac{9}{32}$

$= 1 + \frac{25}{32}$

$\text{Sum} = 21 + 2 + \frac{44}{32} = 24\frac{12}{32} = 24\frac{3}{8}$

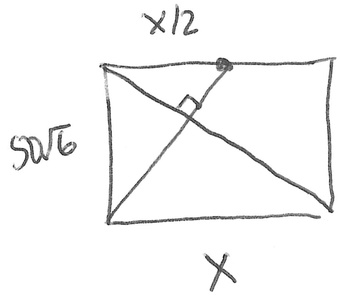
$a_{10} = 1 + \frac{19}{32}$

16) $\boxed{540 cm^3}$ Prism is $3 \times 7 \times 10$

$5 \times 9 \times 12 =$

$60 \times 9 = 540$

17) $\boxed{100\sqrt{3}}$



$\sqrt{x^2 + 15000}$

ratios $50\sqrt{6} : x : hyp$

$\frac{x}{2} : 50\sqrt{6}$

$$\frac{50\sqrt{6}}{\frac{x}{2}} = \frac{x}{50\sqrt{6}}$$

$$\frac{x^2}{2} = 15000$$

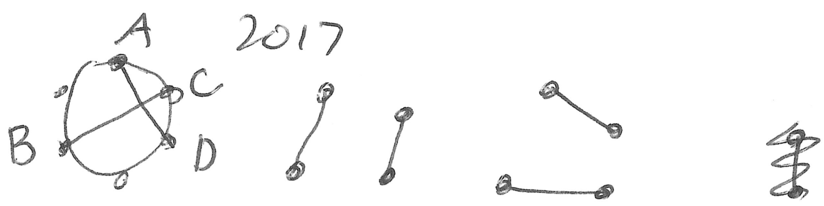
$$x^2 = 30,000$$

$$x = 100\sqrt{3}$$

18) $\boxed{5}$

$\frac{\Delta \Sigma x}{n} = 2 \quad \frac{10}{n} = 2, \ n = 5$

19) $\boxed{\frac{1}{3}}$



if B or C no intersect

if D, yes

20)

$\lfloor 4x+1 \rfloor = 3x - \frac{1}{3}$] must be int

$x = y + \frac{1}{9}$, $y = int$

$\lfloor 4(y+\frac{1}{9})+1 \rfloor = 3(y+\frac{1}{9}) - \frac{1}{3}$

$\lfloor 4y + \frac{13}{9} \rfloor = 3y$

$\frac{1}{3}, \frac{2}{3}$ or int

y must end in $\frac{1}{3}$ or $\frac{2}{3}$

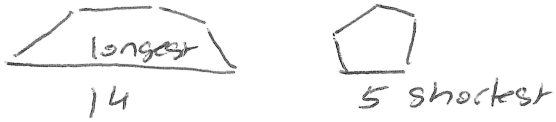
21) $\boxed{\frac{66}{125}}$

Sample Space $= 5^3 = 125$ → 2 odd

$p(ab = \text{odd}) = \frac{9}{25} \times p(c = \text{even}) = \frac{2}{5}$ $\quad \frac{18}{125}$

$p(ab = \text{even}) = \underbrace{\frac{16}{25}}_{(1-\frac{9}{25})} \times p(c = \text{odd}) = \frac{3}{5}$ $\quad + \frac{48}{125} = \boxed{\frac{66}{125}}$

22) $\boxed{10}$



$14 - 5 + 1 = 10$

23) $\boxed{\frac{3}{4}}$

$\binom{4}{3} p^3(1-p) = 9(1-p)^3 p \binom{4}{1}$ ← oops

$p^2 = 9(1-p)^2$

$\left(\frac{3}{4}\right)^2 = 9\left(\frac{1}{4}\right)^2$

} can see 3:1 ratio $p : 1-p$

or solve quadratic...

24) $\boxed{18}$

Sum of 1st n odds is n^2.

25) $\boxed{10}$

$a+b=7,\ d+e=34$

$\underbrace{4(a+b+c+d+e)}_{\text{all 10 combos}} = 7+12+15+17+20+21+24+25+29+34$

$= 204$

$a+b+c+d+e = \frac{204}{4} = 51 \Rightarrow c = \underset{a+b+c+d+e}{51} - \underset{a+b}{7} - \underset{d+e}{34} = \boxed{10}$

20) $\boxed{-\frac{8}{3}}$

$\lfloor 4x+1 \rfloor = 3x - \frac{1}{3}$ RHS $\in \mathbb{Z}$ when $x = \frac{1}{9}, -\frac{2}{9}, -\frac{5}{9}$, etc.

Notice when $x = 0$, LHS > RHS and LHS grows faster so all solutions with $x < 0$.

Try $x = -\frac{2}{9}$, LHS $= 0$, RHS $= -1$

$x = -\frac{5}{9}$ LHS $= -2$, RHS $= -2$

$x = -\frac{8}{9}$ LHS $= -3$, RHS $= -3$

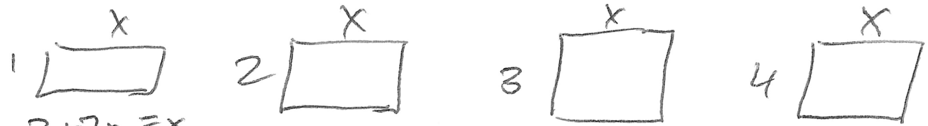
$x = -\frac{11}{9}$ LHS $= -4$, RHS $= -4$

but $x = -\frac{14}{9}$, LHS $= -6$, RHS $= -5$

no more sol

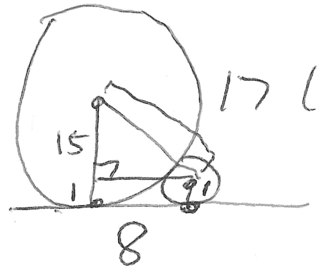
Sum $= -\frac{5}{9} - \frac{8}{9} - \frac{11}{9} = -\frac{24}{9} = \boxed{-\frac{8}{3}}$

26) $\boxed{16}$



1	2	3	4
$2+2x=x$	$4+2x=2x$	$6+2x=3x$	$8+2x=4x$
no	no	$x=6$	$x=4$
		$3\times6=18$	$4\times4=16$ ✓
			better

27) $\sqrt{65}$



17 (16+1)

$17^2-15^2 = 8^2$

so $EF = 8$

$FC = 1$

so $EC = \sqrt{8^2+1^2}$

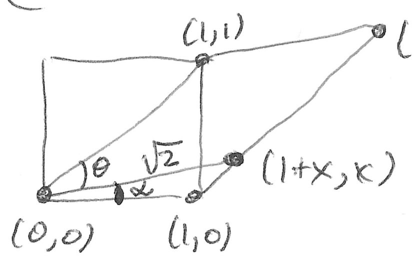
$= \sqrt{65}$

28) $\boxed{\sqrt[3]{2}+\sqrt[3]{3}}$ $\dfrac{5}{(\sqrt[3]{4}-\sqrt[3]{6}+\sqrt[3]{9})} \times \dfrac{(\sqrt[3]{\cancel{2}}+\sqrt[3]{3})}{(\sqrt[3]{2}+\sqrt[3]{3})} = \dfrac{5}{5}(\sqrt[3]{2}+\sqrt[3]{3})$

Use $(x+y)(x^2-xy+y^2) = x^3+y^3$

$(\sqrt[3]{2}+\sqrt[3]{3})(\sqrt[3]{4}-\sqrt[3]{6}+\sqrt[3]{9}) = 2^{\frac{3}{3}}+3^{\frac{3}{3}} = 5$

29) $\boxed{30^\circ}$



because slope = 1 parallel to $y=x$

$\theta+\alpha = 45^\circ$

$x^2+(1+x)^2 = 2$

$x^2+1+2x+x^2 = 2$

$2x^2+2x-1 = 0$

$x = \dfrac{-2\pm\sqrt{4+4(2)}}{4} = \dfrac{-1+\sqrt{3}}{2}$

$\tan\alpha = \dfrac{\frac{-1+\sqrt{3}}{2}}{\frac{+1+\sqrt{3}}{2}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$= \dfrac{\sqrt{3}-1}{\sqrt{3}+1}\times\dfrac{\sqrt{3}-1}{\sqrt{3}-1}$

$= \dfrac{4-2\sqrt{3}}{2}$

$= \boxed{2-\sqrt{3}}$

$\tan\theta = \tan(45-\alpha)$

$= \dfrac{1-\tan\alpha}{1+\tan\alpha} = \dfrac{1-(2-\sqrt{3})}{1+(2-\sqrt{3})}$

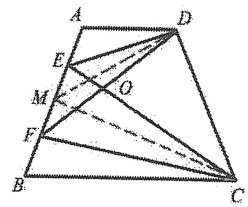
$= \dfrac{\sqrt{3}-1}{3-\sqrt{3}} = \dfrac{\sqrt{3}-1}{\sqrt{3}(\sqrt{3}-1)} = \dfrac{1}{\sqrt{3}} = \boxed{\dfrac{\sqrt{3}}{3}}$

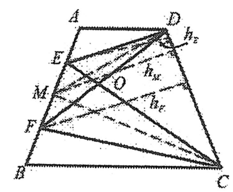
Thus, $\theta = 30^\circ$ since $\tan\boxed{30^\circ} = \dfrac{\sqrt{3}}{3}$

30. Solution: 24 units.

Let M be the midpoint of AB. Since $AE = BF$, $FM = EM$.

Let the heights on CD of triangles ECD, MCD, and FCD be h_E, h_M, and h_F, respectively.





We know that M is the midpoint of EF, $h_M = \frac{1}{2}(h_E + h_F)$.

Thus $S_{\Delta MCD} = \frac{1}{2}(S_{\Delta ECD} + S_{\Delta FCD})$

$\Rightarrow \quad S_{\Delta ECD} + S_{\Delta FCD} = 2S_{\Delta MCD} \qquad (1)$

We also know that $S_{ABCD} = 2S_{\Delta MCD}$. So (1) becomes: $S_{\Delta ECD} + S_{\Delta FCD} = 68$.

The number of square units in the shaded area is

$S = S_{\Delta ECD} + S_{\Delta FCD} - 2S_{\Delta OCD} = 68 - 2 \times 22 = 24$.