Orlando Math Circle AMC Prep 9/8/2023 – Solutions

Algebra

1) If m and n are different positive integers, and if $\frac{\frac{1}{m}-\frac{1}{n}}{1-\frac{1}{mn}} = 1$, what is the value of m?

Solution

$$\frac{\frac{1}{m}-\frac{1}{n}}{1-\frac{1}{mn}} = 1$$

$$\frac{\frac{n}{mn}-\frac{m}{mn}}{\frac{mn}{mn}-\frac{1}{mn}} = 1$$

$$\frac{\frac{n-m}{mn}}{\frac{mn-1}{mn}} = 1$$

$$\frac{n-m}{mn-1} = 1$$

$$nm - 1 = n - m$$

$$nm - n + m - 1 = 0$$

$$(n+1)(m-1) = 0$$

Since n = -1 is false (n must be a positive integer), it follows that **m = 1.**

2) If x is a real number for which $\frac{1}{x^3-3x^2+7x-5} = \frac{5}{6}$, what is the value of $\frac{1}{x^3-3x^2+7x-4}$?

Solution

Flip the first equation to get $x^3 - 3x^2 + 7x - 5 = \frac{6}{5}$. Add 1 to both sides of this equation to get:

$$x^3 - 3x^2 + 7x - 5 + 1 = \frac{6}{5} + 1 = \frac{11}{5}$$

Since $x^3 - 3x^2 + 7x - 4 = \frac{11}{5}$, it follows that $\frac{1}{x^3-3x^2+7x-4} = \mathbf{\frac{5}{11}}$.

3) If $x = t^2 - 4t$ and $y = \frac{1}{t-2}$, what is the value of x when $y = \frac{1}{3\sqrt{2}}$?

Solution
Substituting the expression for y into the last equation, we get:

$$\frac{1}{t-2} = \frac{1}{3\sqrt{2}}$$

It follows that $t - 2 = 3\sqrt{2}$. We must solve for x:

$$x = t^2 - 4t = t(t-4) = (3\sqrt{2} + 2)(3\sqrt{2} - 2) = 18 - 4 = \mathbf{14}$$

Alternatively, notice that:

$$x + 4 = t^2 - 4t + 4 = (t-2)^2 = (3\sqrt{2})^2 = 18$$

so **x = 14.**

Counting

1) How many ordered triplets of integers (a, b, c) are there where 0 < a < b < c < 15?

Solution
Each combination of 3 integers from the set {1, 2, 3, …, 14} represent a single ordered triplet to count. It follows that there are $\binom{14}{3} = \frac{14!}{3!11!} = \frac{14 \times 13 \times 12}{6} = 14 \times 13 \times 2 = \mathbf{364}$ valid ordered triples.

2) How many integers in between 1 and 1000 are divisible by 2, 3 or 5?

<u>Solution</u>

In general, from 1 to n, there are $\left\lfloor \frac{n}{p} \right\rfloor$ integers divisible by a positive integer p. (This is just the largest integer that is less than or equal to the real number division shown. Formally, it's called the floor function.)

It follows that are $\left\lfloor \frac{1000}{2} \right\rfloor = 500$ integers are divisible by 2 in the given range.
It follows that are $\left\lfloor \frac{1000}{3} \right\rfloor = 333$ integers are divisible by 3 in the given range.
It follows that are $\left\lfloor \frac{1000}{5} \right\rfloor = 200$ integers are divisible by 3 in the given range.

But in these separate counts, we've counted some integers twice and some three times. We've counted multiples of 2 x 3 = 6, 2 x 5 = 10 and 3 x 5 = 15 twice, and counted multiples of 2 x 3 x 5 = 30 three times.

Let's subtract out the multiples of 6, 10 and 15 that were double counted. (Thus, we must subtract out $\left\lfloor \frac{1000}{6} \right\rfloor = 166$, $\left\lfloor \frac{1000}{10} \right\rfloor = 100$, and $\left\lfloor \frac{1000}{15} \right\rfloor = 66$.

But when we do this, we have subtracted out the multiples of 30 three times and not counted them at all. Thus, we must add these back in: $\left\lfloor \frac{1000}{30} \right\rfloor = 33$.

Our final tally is 500 + 333 + 200 – 166 – 100 – 66 + 33 = <u>**734.**</u>

3) An animal shelter has two cats: Felix and Flopsy and three dogs: Roger, Amelia and Coco. All of these animals are to be adopted by a potential four owners: Monty, Nellie, Ophelia and Paul. Assuming that no owner receives both a cat and a dog and that it's not necessary for each potential owner to receive a pet, in how many ways can the five pets be distributed amongst the owners? (Two arrangements are different if any owner-pet combo is different between the two arrangements. For example, if Monty owns Felix in one arrangement but Nellie owns her in another, those two are different arrangements.)

Solution

Approach #1
Either one person gets both cats, or two different people get 1 cat each. Let's split up our work into those two cases:

Case 1: 1 person gets 2 cats → we pick the owner of the 2 cats for which there are 4 possibilities. Now, let's assign each of the dogs. Each dog must be assigned to one of 3 owners, thus we can assign the dogs in 3 x 3 x 3 = 27 ways. Thus, for this case, there are a total of 4 x 27 = 108 arrangements.

Case 2: 2 different people get 1 cat each → we pick the owner of Felix in 4 ways, and the owner of Flopsy in 3 ways (can't go to the owner of Felix). Now, all three dogs must be assigned to one of two owners. Thus, we have 2 choices for each dog. Thus the dogs can be given to the owners in 2 x 2 x 2 = 8 ways. It follows that there are 4 x 3 x 8 = 96 arrangements.

Thus, the pets can be assigned owners I 108 + 96 = **204 ways.**

Approach #2
This approach was suggested by Deron during the session.

We can line up the owners in 4! = 24 ways.

Now, consider combinations of # of dogs and cats. (To avoid overcounting, we will list the dogs before the cats an more of an animal before less of an animal.)

1 D, 1 D, 1 D, 2 C	(this occurs in 1 way)
2 D, 1 D, 2 C, none	(we choose the 2 dogs in 3 ways, so this occurs in 3 ways)
2 D, 1 D, 1 C, 1 C	(we choose the 2 dogs in 3 ways, so this occurs in 3 ways)
3 D, 1 C, 1 C, none	(this occurs in 1 way)

Notice that in some cases here we had to list none as what the last owner receives.

There is one combination missing here (3D, 2C), but that is for a reason. For the 8 combinations of cats and dogs above, we can assign each to owners in 4! ways, resulting in

24 x 8 = 192 assignments of pets to owners.

Now, let's handle the last case of 3 dogs to one owner and 2 cats to another owner. We can choose the owner of the dogs in 4 ways and the owner of the cats in 3 ways, for a total of 4 x 3 = 12 ways.

Our final answer is 192 + 12 = **204 ways** to assign pets to owners.

Number Theory

1) p and q are distinct primes and 50! is divisible by both p and q. What is the maximum possible value of pq?

Solution

The largest prime that divides into 50! is 47. The second largest prime that divides into 50! is 43. (We can see this just by writing out the long form of the product of 50!.) It follows that the maximum possible value of pq = 47 x 43 = **2021.**

2) n is a positive integer with exactly 15 divisors. What is the smallest possible value of n?

Solution

Given the prime factorization of an integer as $n = p^a q^b$, the integer has exactly (a+1)(b+1) divisors. Since 15 = 5 x 3, if n has two distinct prime factors, a = 4 and b = 2 is required. If we want to minimize n, we want to choose p to be the smallest prime and q to be the next smallest. Thus, one candidate is $2^4 \times 3^2 = 144$. The only other candidate one might consider is of the form $n = p^a$ which has a+1 divisors. But, it's easy to see that 2^{14} is larger than 144. It follows that the smallest positive integer with exactly 15 divisors is **144.**

3) Define f(n) to be the least common multiple of the positive integers 1 through n. For example f(4) = 12 and f(5) = 60. Under what circumstances does f(n) = f(n – 1)?

Solution

Consider the involvement of a single prime, p, in the value of f(n). If $p^k <= n$, for some positive integer k, then it follows that $p^k \mid f(n)$, by definition of LCM.

If for that same integer k we have $p^{k+1} > n$, then since no individual integer from 1 to n has p^{k+1} in its prime factorization, it follows that p^{k+1} does NOT divide evenly into f(n).
Thus, if $n = p^k$, for any prime p and positive integer, k, it follows that $f(n) \neq f(n-1)$. In fact, using the observation above, we see that in this case $f(n) = pf(n-1)$.

In all other cases, n must have at least 2 distinct prime numbers in its prime factorization. In all of these cases, we can see that each individual prime raised to a power in the prime factorization is less than n and must appear as a term in the items for which the LCM is computed. Thus, each of these terms must be a divisor of f(n).

It follows that f(n) = f(n – 1) if and only if n has at least 2 distinct prime factors.

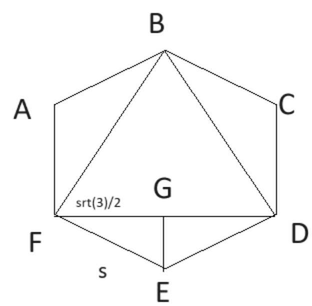
Geometry

1) ABCDEF is a regular hexagon. What is the ratio of the area of BDF (an equilateral triangle formed with alternating vertices of the hexagon) to the area of ABCDEF?

Solution

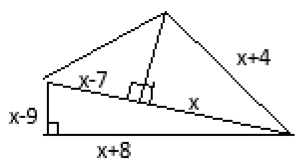
A regular hexagon with side length s can be subdivided into 6 equilateral triangles of side length s. This means the area of the whole hexagon is $6\left(\frac{s^2\sqrt{3}}{4}\right) = \frac{3s^2\sqrt{3}}{2}$.

BDF is also an equilateral triangle. Using the diagram below and two 30-60-90 triangles (GEF and GDE), we can determine that BD = $s\sqrt{3}$.



It follows that the area of BDF is $\frac{(\sqrt{3}s)^2\sqrt{3}}{4} = \frac{s^2 3\sqrt{3}}{4}$. The desired ratio is $\frac{s^2 3\sqrt{3}}{4} \Big/ \frac{s^2 3\sqrt{3}}{2} = \mathbf{\frac{1}{2}}$

2) In the quadrilateral shown, one of the diagonals is drawn, some of the line segments are marked with their lengths, and some line segments are marked as being perpendicular. (There are three right angled triangles in the picture.) What is the perimeter of the quadrilateral?



<u>Solution</u>

Use the bottom right triangle to set up the Pythagorean Theorem:

$$(x-9)^2 + (x+8)^2 = (2x-7)^2$$

$$x^2 - 18x + 81 + x^2 + 16x + 64 = 4x^2 - 28x + 49$$

$$2x^2 - 26x - 96 = 0$$

$$x^2 - 13x - 48 = 0$$

$$(x-16)(x+3) = 0$$

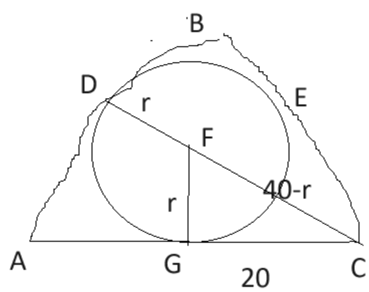
Thus, x = 16. The bottom right triangle is a 7-24-25 triangle. The top right right triangle is a 12-16-20 and the top left right triangle is a 9-12-15 triangle.

It follows that the desired perimeter is 24 + 7 + 15 + 20 = **<u>66.</u>**

3) Equilateral Gothic arch ABC is made by drawing line segment AC, circular arc AB with center C, and circular arc BC with center A. A circle inscribed in this Gothic arch is tangent to arc AB, arc BC and line segment AC. If AC = 40, what is the area of the circle?

Solution

Let the circle intersect the arc AB and D and the arch BC and E. Let the center of the circle be F. Let the circle touch AC at point G. Because the circle hits the arc at only one spot, the tangent line to the circle and arc at that point is the same, meaning that the line segment CD is a radius of the circle that arc AB is a part of and this line segment must go through F. This means that CF = 40 – r, where r is the radius of the circle. Then we can use right triangle CFG to solve for r. Here is the picture (I apologize for the quality of the picture. I am really bad at using Paint.):



Now, setting up the Pythagorean Theorem on the right triangle CGF we have

$$20^2 + r^2 = (40 - r)^2$$

$$400 + r^2 = 1600 - 80r + r^2$$

$$80r = 1200$$

$$r = 15$$

It follows that the area of the circle is **225π.**

Logs

1) Determine the value of x > 1 for which $log_2(2+x) = log_x 2 + log_x x$.

Solution

$$log_2(2+x) = log_x 2 + log_x x$$

$$log_2(2+x) = log_x 2 + 1$$

Take 2 to the power of each side of this equation to yield:

$$2^{log_2(2+x)} = 2^{log_x 2+1}$$

$$2 + x = (2^{log_x 2})(2^1)$$

$$2 + x = 2x$$

$$\mathbf{x = 2}$$

The second to last step is because the exponent and log are inverse functions of each other.

2) Find both values of x which satisfy the following equation:

$$\frac{log_2 x}{log_4 2x} = \frac{log_8 4x}{log_{16} 8x}$$

Solution

Note that $log_4 x = \frac{log_2 x}{log_2 4} = \frac{1}{2} log_2 x$. Similarly, $log_8 x = \frac{1}{3} log_2 x$ and $log_{16} x = \frac{1}{4} log_2$.

Use the log addition rule:

$$\frac{log_2 x}{log_4 2 + \ log_4 x} = \frac{log_8 4 + log_8 x}{log_{16} 8 + log_{16} x}$$

Now, let $A = log_2 x$:

$$\frac{A}{\frac{1}{2} + \frac{A}{2}} = \frac{\frac{2}{3} + \frac{A}{3}}{\frac{3}{4} + \frac{A}{4}}$$

Cross multiply:

$$A\left(\frac{A+3}{4}\right) = \left(\frac{A+1}{2}\right)\left(\frac{A+2}{3}\right)$$

$$3A(A+3) = 2(A+1)(A+2)$$

$$3A^2 + 9A = 2A^2 + 6A + 4$$

$$A^2 + 3A - 4 = 0$$

$$(A+4)(A-1) = 0$$

Thus, A = -4 or A = 1, which means that $\boldsymbol{x = \frac{1}{16}}$ **or** $\boldsymbol{x = 2}$.

3) Define a function f as follows: f(1) = 2. For all integers n > 1, $f(n) = (f(n-1))^{2^n}$.

What is the value of $log_{65536}(log_{65536}(f(4))$?

Express your answer in the form $2^a + 2^b$, where both a and b are integers. Note that $65536 = 2^{16}$.

<u>Solution</u>
First, let's calculate f(4). $f(2) = f(1)^4 = 2^4$, $f(3) = (f(2))^8 = (2^4)^8 = 2^{32}$, $f(4) = f(3)^{16} = (2^{32})^{16} = 2^{512}$.

Let X = $log_{65536}(f(4))$. Then, using the power rule, we have X = $512 log_{65536} 2$. Use the log base change rule with base 2 to get the following:

$$log_{65536} 2 = \frac{log_2 2}{log_2 65536} = \frac{1}{16}$$

Thus, X = $\frac{512}{16} = 32$.

The value we desire is = $log_{65536} X = log_{65536} 32 = \frac{log_2 32}{log_2 65536} = \frac{5}{16} = \mathbf{2^{-2} + 2^{-4}}$.