

Given 1/10/23

TRIG HMK SOLS PROBS: 8, 10, 12, 25, 26, 33, 34, 37

~~missing: 25e, 26b and 37~~

8) Using the Extended Law of Sines,

$$\frac{\sin A}{a} = \frac{1}{2R} \Rightarrow a = 2R \sin A, b = 2R \sin B, c = 2R \sin C$$

$$\frac{a}{b+c} = \frac{2R \sin A}{2R \sin B + 2R \sin C} = \frac{2R}{2R} \times \frac{\sin A}{(\sin B + \sin C)}$$

Now, look at $\frac{\sin A}{\sin B + \sin C}$?

$$\begin{aligned} \frac{\sin A}{\sin B + \sin C} &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \left(\frac{B+C}{2} \right) \cos \left(\frac{B-C}{2} \right)} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos \left(90^\circ - \frac{B+C}{2} \right) \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos \left(\frac{180 - B - C}{2} \right) \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos \frac{A}{2} \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{\sin \frac{A}{2}}{\cos \left(\frac{B-C}{2} \right)} \geq \sin \frac{A}{2} \end{aligned}$$

Since $-180^\circ < B-C < 180^\circ$, $-90 < \frac{B-C}{2} < 90$, so
 $0 < \cos \left(\frac{B-C}{2} \right) < 1$.

$$\text{Thus, } \frac{a}{b+c} = \frac{\sin A}{\sin B + \sin C} = \frac{\sin \frac{A}{2}}{\cos \left(\frac{B-C}{2} \right)} \geq \sin \frac{A}{2}$$

Proving the assertion that $\sin \frac{A}{2} \leq \frac{a}{b+c}$.

10) Let $S = \sin x$, $C = \cos x$. $S^2 + C^2 = 1$.

$$1 = (C^2 + S^2)^2 = C^4 + S^4 + 2S^2C^2 \Rightarrow C^4 + S^4 = 1 - 2S^2C^2$$

$$(S^4 + C^4)(S^2 + C^2) = S^6 + C^6 + S^2C^4 + S^4C^2$$

$$(S^4 + C^4)(S^2 + C^2) = S^6 + C^6 + S^2C^2(S^2 + C^2)$$

$$\Rightarrow S^4 + C^4 = S^6 + C^6 + S^2C^2, \text{ since } S^2 + C^2 = 1$$

$$\Rightarrow S^6 + C^6 = S^4 + C^4 - S^2C^2 = 1 - 2S^2C^2 - S^2C^2 = \boxed{1 - 3S^2C^2}$$

$$f_4(x) - f_6(x) = \frac{1}{4}(1 - 2S^2C^2) - \frac{1}{6}(1 - 3S^2C^2)$$

$$= \frac{1}{4} - \frac{1}{2}S^2C^2 - \frac{1}{6} + \frac{1}{2}S^2C^2$$

$$= \frac{1}{4} - \frac{1}{6} = \boxed{\frac{1}{12}}$$

12) $(3\sin A + 4\cos B)^2 = 6^2$

$$+ (3\cos A + 4\sin B)^2 = (1)^2$$

$$\boxed{9\sin^2 A + 24\sin A \cos B + 16\cos^2 B + 9\cos^2 A + 24\cos A \sin B + 16\sin^2 B} = 37$$

$$9 + 24(\sin A \cos B + \cos A \sin B) + 16 = 37$$

$$24 \sin(A+B) = 12$$

$$\sin(A+B) = \frac{1}{2}$$

So $A+B = 30^\circ$ or $A+B = 150^\circ$.

Note that if $A \leq 30^\circ$, then $4\cos B \geq 6 - 3\sin 30^\circ$
 $4\cos B \geq 6 - \frac{3}{2} = \frac{9}{2}$

but this is impossible since $4\cos B \leq 4$.

It follows that $A+B = 150^\circ$, so

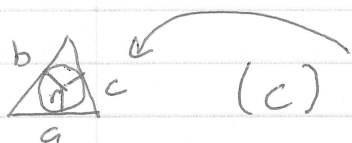
$$\boxed{C = 30^\circ}$$

25) (a) By Extended Law of Sines $\frac{a}{\sin A} = 2R$, so

$$4R = \frac{2a}{\sin A} \times \frac{\frac{1}{2}bc}{\frac{1}{2}bc} = \frac{abc}{\frac{1}{2}bc \sin A} = \frac{abc}{[ABC]}.$$

$$(b) [ABC] = \frac{1}{2}bc \sin A = \frac{1}{2} \cdot 2R \sin B \cdot 2R \sin C \cdot \sin A, \text{ via ext. law sine}$$

$$= \boxed{2R^2 \sin A \sin B \sin C}$$



(c) Recall that $[ABC] = rs = r\left(\frac{a+b+c}{2}\right)$

$$[ABC] = r\left(\frac{a+b+c}{2}\right) \quad \text{note } \frac{a}{\sin A} = 2R$$

$$2[ABC] = r(a+b+c), \quad \text{so } a = 2R \sin A$$

$$2 \cdot \frac{abc}{4R} = r(a+b+c), \text{ from part (a).}$$

$$\frac{2R \sin A \cdot 2R \sin B \cdot 2R \sin C}{2R} = r(2R \sin A + 2R \sin B + 2R \sin C)$$

$$\cancel{2R}(\cancel{2R}) \sin A \sin B \sin C = \cancel{2R} \cdot r(\sin A + \sin B + \sin C)$$

$$\boxed{2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)}$$

25d) By Law of Cosines $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\begin{aligned}\sin^2 \frac{A}{2} &= \frac{1 - \cos A}{2} = \frac{1}{2} - \frac{b^2 + c^2 - a^2}{4bc} \\ &= \frac{2bc - b^2 - c^2 + a^2}{4bc} = \frac{a^2 - (b^2 - 2bc + c^2)}{4bc} \\ &= \frac{a^2 - (b-c)^2}{4bc} = \frac{(a+b-c)(a-b+c)}{4bc}\end{aligned}$$

Since $s = \frac{a+b+c}{2}$, $\sin^2 \frac{A}{2} = \frac{(s-c)(s-b)}{bc}$, thus

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \frac{(s-a)(s-b)(s-c)}{abc} \times \frac{s}{s}$$

$$= \frac{[ABC] [ABC]}{s \times 4R [ABC]}$$

$$= \frac{r}{4R}$$

Thus $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R}$, now multiply both sides by $4R$

$$\boxed{r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}, \text{ as desired}$$

25e) Prove $a \cos A + b \cos B + c \cos C = \frac{abc}{2R^2}$

$$\begin{aligned} a \cos A + b \cos B + c \cos C &= 2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C \\ &= R (\sin 2A + \sin 2B + \sin 2C) \\ &= R (\sin 2A + \sin 2B + \sin (360 - (2A + 2B))) \\ &= R (\sin 2A + \sin 2B - \sin (2A + 2B)) \\ &= R (\sin 2A + \sin 2B - \sin 2A \cos 2B - \cos 2A \sin 2B) \\ &= R [\sin 2A (1 - \cos 2B) + \sin 2B (1 - \cos 2A)] \\ &= R [\sin 2A \cdot 2 \sin^2 B + \sin 2B \cdot 2 \sin^2 A] \\ &= 2R [2 \sin A \cos A \sin^2 B + 2 \sin B \cos B \sin^2 A] \\ &= 4R \sin A \sin B [\cos A \sin B + \sin A \cos B] \\ &= 4R \sin A \sin B \sin (A+B) \\ &= 4R \sin A \sin B \sin (180 - (A+B)) \\ &= 4R \sin A \sin B \sin C \\ &= 4R \cdot \frac{[ABC]}{2R^2}, \text{ from } 25(b). \\ &= \frac{4R}{2R^2} \cdot \frac{abc}{4R}, \text{ from } 25(a). \\ &= \frac{abc}{2R^2} \quad \checkmark \end{aligned}$$

$$26a) [ABC] = rs \Rightarrow S = \frac{[ABC]}{r}$$

$$\text{From 25b) } [ABC] = 2R^2 \sin A \sin B \sin C$$

$$\Rightarrow S = \frac{2R^2 \sin A \sin B \sin C}{r}$$

$$\text{From 25d) } r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\Rightarrow S = \frac{2R^2 \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2} \cdot 2 \sin \frac{B}{2} \cos \frac{B}{2} \cdot 2 \sin \frac{C}{2} \cos \frac{C}{2}}{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}$$

$$\boxed{S = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

26b) First, we prove, via Extended law of sines that

$$\sin \frac{A}{2} \leq \frac{a}{b+c}; \quad \frac{a}{b+c} = \frac{\sin A}{\sin B + \sin C}, \text{ via extended law of sines}$$

$$\begin{aligned} \frac{a}{b+c} &= \frac{\sin A}{\sin B + \sin C} = \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \left(\frac{B+C}{2} \right) \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \left(\frac{B+C}{2} \right) \cos \left(\frac{B-C}{2} \right)} \end{aligned}$$

Thus,

$$\frac{a}{b+c} \geq \sin \frac{A}{2}$$

Similarly

$$\frac{b}{a+c} \geq \sin \frac{B}{2}$$

and

$$\frac{c}{a+b} \geq \sin \frac{C}{2}$$

$$\begin{aligned} &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \left(\frac{B+C}{2} \right) \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \left(\frac{180-A}{2} \right) \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{\sin \frac{A}{2} \cos \frac{A}{2}}{\sin \left(90 - \frac{A}{2} \right) \cos \left(\frac{B-C}{2} \right)} \\ &= \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} \times \frac{\cos \frac{A}{2}}{\cos \left(\frac{B-C}{2} \right)} \\ &= \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}} \geq \sin \left(\frac{A}{2} \right) \end{aligned}$$

26b cont)

$$\text{Multiply to get } \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{abc}{(a+b)(a+c)(b+c)}$$

Using AM/GM inequality $a+b \geq 2\sqrt{ab}$, so

$$\begin{aligned}(a+b)(a+c)(b+c) &\geq 2\sqrt{ab} \cdot 2\sqrt{ac} \cdot 2\sqrt{bc} \\ &= 8\sqrt{a^2b^2c^2} \\ &= 8abc\end{aligned}$$

$$\text{Thus, } \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{abc}{(a+b)(a+c)(b+c)} \leq \frac{abc}{8abc} = \frac{1}{8}$$

* Result w/o proof: In $\triangle ABC$,

$$\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} + 2\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = 1$$

$$\Rightarrow \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \geq 1 - 2\left(\frac{1}{8}\right) = \frac{3}{4}$$

$$\Rightarrow 1 - \cos^2 \frac{A}{2} + 1 - \cos^2 \frac{B}{2} + 1 - \cos^2 \frac{C}{2} \geq \frac{3}{4}$$

$$\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \leq \frac{9}{4}$$

Using AM/GM w/3 terms,

$$\frac{9}{4} \geq \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \geq 3 \sqrt[3]{\cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$\Rightarrow \frac{3}{4} \geq \sqrt[3]{\cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2}}$$

$$\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \leq \left(\frac{3}{4}\right)^{3/2} = \frac{3\sqrt{3}}{8}$$

From part (A),

$$s = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \leq 4R \cdot \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}R}{2} \checkmark$$

33) From 25d,

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$r = 4(81) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\frac{1}{32} = \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Note $\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right) = 2 \sin \frac{A}{2} \sin \frac{B}{2}$

$$\frac{1}{32} = \left(\frac{1}{2} \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right) \right) \sin \frac{C}{2}$$

$$\frac{1}{16} = \left(\cos \frac{120^\circ}{2} - \cos \left(\frac{180}{2} - \frac{(A+B)}{2} \right) \right) \sin \frac{C}{2}$$

$$\frac{1}{16} = \left(\frac{1}{2} - \sin \frac{C}{2} \right) \sin \frac{C}{2}, \text{ since } A+B+C=180^\circ$$

$$\frac{1}{16} = \frac{1}{2} \sin \frac{C}{2} - \sin^2 \frac{C}{2}$$

$$\sin^2 \frac{C}{2} - \frac{1}{2} \sin \frac{C}{2} + \frac{1}{16} = 0$$

$$\left(\sin \frac{C}{2} - \frac{1}{4} \right)^2 = 0$$

$$\sin \frac{C}{2} = \frac{1}{4} \Rightarrow \cos \frac{C}{2} = \frac{\sqrt{15}}{4}, \text{ since } C < 180^\circ$$

$$\cos C = 2 \cos^2 \frac{C}{2} - 1$$

$$= 2 \left(\frac{15}{16} \right) - 1 = \frac{30-16}{16} = \frac{14}{16} = \boxed{\frac{7}{8}}$$

34) By Extended Law of Sines, $a = 2R \sin A$

$$\frac{a-b}{a+b} = \frac{2R \sin A - 2R \sin B}{2R \sin A + 2R \sin B}$$

$$= \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{\sin\left(\frac{A+B}{2} + \left(\frac{A-B}{2}\right)\right) - \sin\left(\frac{A+B}{2} - \left(\frac{A-B}{2}\right)\right)}{\sin\left(\frac{A+B}{2} + \left(\frac{A-B}{2}\right)\right) + \sin\left(\frac{A+B}{2} - \left(\frac{A-B}{2}\right)\right)}$$

$$= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}$$

$$= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}$$

$$= \cot\left(\frac{A+B}{2}\right) \tan\left(\frac{A-B}{2}\right)$$

$$= \tan\left(\frac{180}{2} - \frac{(A+B)}{2}\right) \tan\left(\frac{A-B}{2}\right)$$

$$= \tan\left(\frac{C}{2}\right) \tan\left(\frac{A-B}{2}\right)$$

$$= \boxed{\tan\left(\frac{A-B}{2}\right) \tan\left(\frac{C}{2}\right)}$$

37) By extended law of sines, $a = 2R \sin A$, etc.
Substituting, we aim to prove:

$$2R \sin A \cos B + 2R \sin B \cos C + 2R \sin C \cos A = \frac{2R \sin A + 2R \sin B + 2R \sin C}{2}$$

$$\cancel{2R} (\sin A \cos B + \sin B \cos C + \sin C \cos A) = R (\sin A + \sin B + \sin C)$$

$$2 \sin A \cos B + 2 \sin B \cos C + 2 \sin C \cos A = \sin A + \sin B + \sin C$$

Let's start with the LHS:

$$\begin{aligned} 2 \sin A \cos B + 2 \sin B \cos C + 2 \sin C \cos A &= \sin(A+B) + \sin(A-B) \\ &\quad + \sin(B+C) + \sin(B-C) \\ &\quad + \sin(C+A) + \sin(C-A) \end{aligned}$$

$$\begin{aligned} \text{Since } \sin x &= \sin(180-x), & &= \sin C + \sin(A-B) \\ & & &+ \sin A + \sin(B-C) \\ & & &+ \sin B + \sin(C-A) \end{aligned}$$

When

If ~~we can show~~ $\sin(A-B) + \sin(B-C) + \sin(C-A) = 0$,
~~we are done~~ is precisely when the original
statement is true. Let's work with this expression
to make it look like a product.

$$\sin(A-B) + \sin(B-C) + \sin(C-A) = 2 \sin \frac{A-C}{2} \cos \frac{A+C-2B}{2} + \sin(C-A)$$

$$= 2 \sin \left(\frac{A-C}{2} \right) \cos \frac{A+C-2B}{2} + 2 \sin \left(\frac{C-A}{2} \right) \cos \left(\frac{C-A}{2} \right)$$

$$= 2 \sin \frac{A-C}{2} \left[\cos \frac{A+C-2B}{2} \cancel{\cos \frac{C-A}{2}} \right]$$

$$= 2 \sin \left(\frac{A-C}{2} \right) \left[2 \cos \left(\frac{C-B}{2} \right) \cos \left(\frac{A-B}{2} \right) \right]$$

The sign flips
to a minus here
because $\sin \left(\frac{C-A}{2} \right) =$
 $-\sin \left(\frac{A-C}{2} \right)$.

$$\begin{aligned}
 37 \text{ cont) } &= 2 \sin\left(\frac{A-C}{2}\right) \left[\cos \frac{A+C-2B}{2} - \cos \frac{C-A}{2} \right] \\
 &= 2 \sin\left(\frac{A-C}{2}\right) \left[2 \sin\left(\frac{C-B}{2}\right) \sin\left(\frac{B-A}{2}\right) \right] \\
 &= 4 \sin\left(\frac{A-C}{2}\right) \sin\left(\frac{C-B}{2}\right) \sin\left(\frac{B-A}{2}\right)
 \end{aligned}$$

In order for this to be 0 with $0 < A, B, C < 180^\circ$ we must have $A-C=0$ or $C-B=0$ or $B-A=0$, implying that the triangle must be isosceles, as desired.