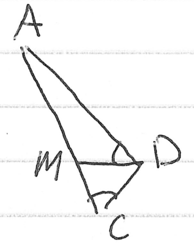
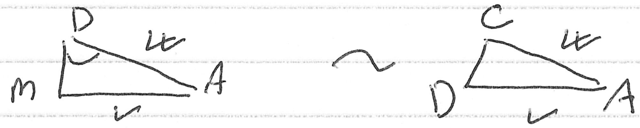
OMC Geometry Hmk Solutions (Due 11/3/22)

167) 

$\measuredangle ACD = \measuredangle MDA$ and
$\measuredangle DAC = \measuredangle DAM$ so
$\triangle MDA \sim \triangle DCA$



So $\dfrac{AM}{AD} = \dfrac{AD}{AC}$

Solving for AD^2 we get

$$\boxed{AD^2 = (AM)(AC)}$$

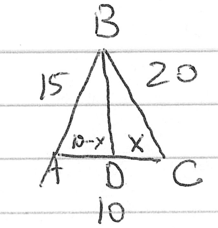
168) Note that if 1 is a side, and
x is the second side, x+1 is too big
for the third side because

$$1 + x \leq (1+x) \text{, triangle inequality}$$

Try $a=2$, then $b=3, c=4$ ✓
or $b=4, c=5$ ✓
but if $b=5, c=6$ and $2+5+6 \geq 13$ ×
Next $a=3, b=4, c=5$ ✓

There are $\boxed{3}$ total solutions.

169) $\boxed{\frac{40}{7}}$



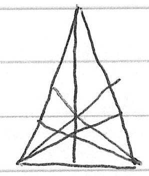
$$\frac{20}{x} = \frac{15}{10-x}$$

$$200 - 20x = 15x$$

$$35x = 200$$

$$x = \frac{40}{7}$$

170) $\boxed{7}$



Three altitudes.

1 altitude = 1 median (to "short" side)

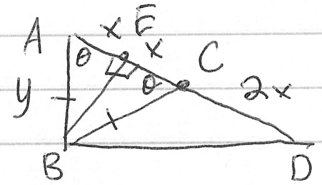
2 "new" medians.

1 angle bisector = altitude (to "short" side)

2 "new" angle bisectors

$3+2+2=7$.

171) $\boxed{60^\circ}$

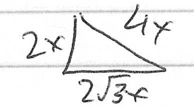


$AEB \sim ABD$

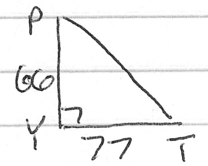
Thus $\frac{x}{y} = \frac{y}{4x}$

$\Rightarrow 4x^2 = y^2 \Rightarrow y = 2x$

Triangle is
30°-60°-90°



172) $\boxed{96}$



$$PT = \sqrt{66^2+77^2} = \sqrt{11^2 \cdot 6^2 + 11^2 \cdot 7^2}$$

$$= \sqrt{11^2(36+49)}$$

$$= = 11\sqrt{85}$$

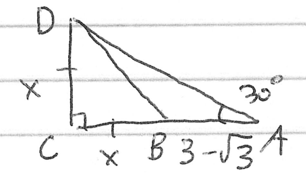
$x+y = 11+85 = \boxed{96}$

173) $\boxed{\cancel{2}}$ $\boxed{\frac{1}{2}}$

$$A = \frac{1}{2}40h_1 = \frac{1}{2}80h_3$$

$$h_1 = 2h_3 \Rightarrow K = \frac{1}{2}$$

174) $\boxed{\frac{3}{2}}$



$$\frac{x+3-\sqrt{3}}{x} = \sqrt{3}$$

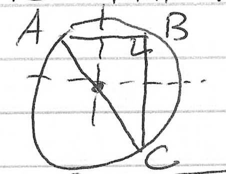
$$x+3-\sqrt{3} = \sqrt{3}x$$

$$x = \frac{3-\sqrt{3}}{\sqrt{3}-1} = \frac{\sqrt{3}(\sqrt{3}-1)}{\sqrt{3}-1}$$

$$= \sqrt{3}$$

$\triangle BCD = \frac{1}{2}\sqrt{3}\sqrt{3} = \frac{3}{2}$

175) The perpendicular bisectors of the sides of a triangle meet at the circumcenter of that triangle, because all 3 pts are equidistant from the circumcenter, since the lines were perpendicular bisectors. This means that the circumcenter IS on the triangle:



Thus, the longest side of the triangle is a diameter and it follows that the angle opposite this side is $\frac{180°}{2} = \boxed{90°}$, thus the triangle must be a right triangle.

176) Recall that $b+c > a$, by the triangle inequality.

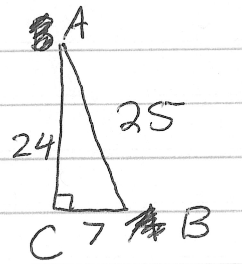
$$2\text{Area} = ah_a = bh_b = ch_c.$$

$$a = \frac{2\text{Area}}{h_a}, \quad b = \frac{2\text{Area}}{h_b}, \quad c = \frac{2\text{Area}}{h_c}$$

$$b+c > a \implies \frac{2\text{Area}}{h_b} + \frac{2\text{Area}}{h_c} > \frac{2\text{Area}}{h_a}$$

$$\implies \frac{1}{h_b} + \frac{1}{h_c} > \frac{1}{h_a} \quad \checkmark$$

177)

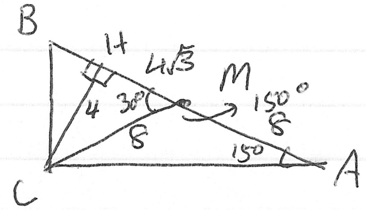


$$AC^2 = AB^2 - BC^2$$
$$AC^2 = 625 - 49$$
$$AC^2 = 576 \implies AC = 24$$

$$\boxed{\sin B = \frac{24}{25}} \quad \boxed{\cos A = \frac{24}{25}}$$

$$\boxed{\cot A = \frac{24}{7}} \quad \boxed{\csc B = \frac{25}{24}}$$

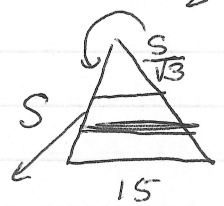
178) $\boxed{32}$



$\measuredangle HMC = 30°$, so $HM = 4\sqrt{3}$ and $MC = 8$.
Since M is median, $MC = AC$, so $AM = 8$

$$\text{Area}(ABC) = \frac{1}{2} \times HC \times AB = \frac{1}{2} \times HC \times (2 \times AM) = \frac{1}{2} \times 4 \times 16 = \boxed{32}$$

179) $\boxed{5\sqrt{6}}$

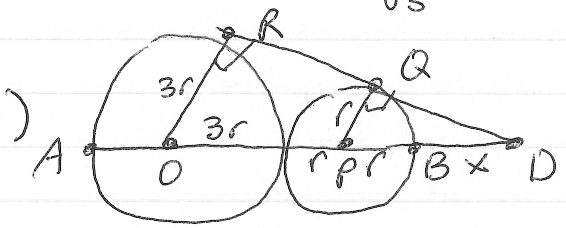


$\frac{\sqrt{2}}{\sqrt{3}}S - \frac{S}{\sqrt{3}}$

If area is $\frac{1}{3}$, side ratio is $\frac{1}{\sqrt{3}}$.
If area is $\frac{2}{3}$, side ratio is $\frac{\sqrt{2}}{\sqrt{3}}$.

$$\text{Second parallel} = \frac{\sqrt{2}}{\sqrt{3}} \times 15 = \frac{\sqrt{2}}{\sqrt{3}} \times 15 \times \frac{\sqrt{3}}{\sqrt{3}} = 5\sqrt{6}$$

180)



Let O = center big circle
P = center small circle
R, Q = pts of tengency.

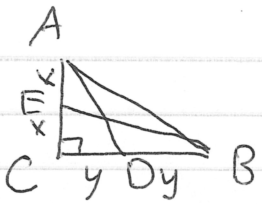
$\triangle DQP \sim \triangle DRO$ with ratio $1:3$

$$\frac{r}{r+x} = \frac{3r}{5r+x}$$

$$5r^2 + rx = 3r^2 + 3rx$$
$$2r^2 = 2rx$$
$$r = x$$

This proves $|BD|$ equals the radius of the smaller circle.

181) $\boxed{\frac{\sqrt{5}}{2} AB}$



Let $x = AE$, $y = CD$, then

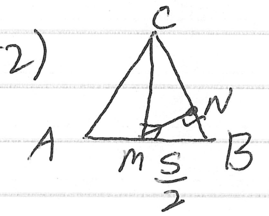
$$(2x)^2 + (2y)^2 = AB^2$$

$$AD^2 = (2x)^2 + y^2$$
$$BE^2 = x^2 + (2y)^2$$

So, $AD^2 + BE^2 = (2x)^2 + y^2 + x^2 + (2y^2)$
$= 5x^2 + 5y^2$
$= \frac{5}{4}\left((2x)^2 + (2y)^2\right)$

Thus $\sqrt{AD^2 + BE^2} = \frac{\sqrt{5}}{2}\sqrt{AB^2} = \frac{\sqrt{5}}{2} AB.$

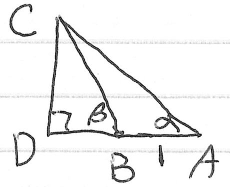
182)



Let $S = BC$.
Note $\triangle BNM$ is $30°$-$60°$-$90°$ tri.
because $\measuredangle MNB = 90°$ and $\measuredangle NBM = 60°$
Thus $BN = \frac{1}{2} BM = \frac{1}{2} \cdot \frac{S}{2} = \frac{S}{4}$

$4BN = 4\left(\frac{S}{4}\right) = S = BC$ ✓

183) $\boxed{CD = \frac{1}{\cot\alpha - \cot\beta}}$



$\tan\beta = \frac{CD}{BD}$, $\tan\alpha = \frac{CD}{1 + BD}$

$\tan\alpha = \frac{CD}{1 + \frac{CD}{\tan\beta}} = \cancel{\frac{CD}{\tan\beta + CD}}$

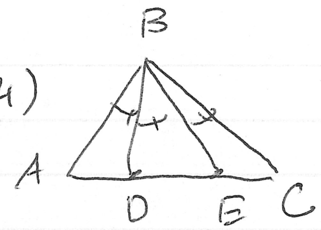
Thus, $1 + \frac{CD}{\tan\beta} = \frac{CD}{\tan\alpha}$

$$\frac{CD}{\tan\alpha} - \frac{CD}{\tan\beta} = 1$$

$$CD(\cot\alpha - \cot\beta) = 1$$

$$CD = \frac{1}{\cot\alpha - \cot\beta}$$

184)



Use angle bisector formula with $\triangle ABE$ and $\triangle DBC$.

(1) $\frac{AB}{AD} = \frac{BE}{DE}$ (use $\triangle ABE$)

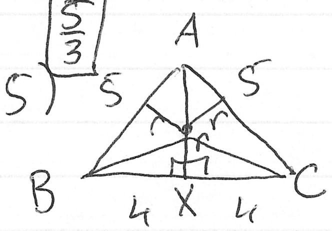
(2) $\frac{BD}{DE} = \frac{BC}{EC}$ (use $\triangle DBC$)

From (1), $AB = \frac{AD \times BE}{DE}$, From (2), $CE = \frac{BC \times DE}{BD}$

$AD = \frac{AB \times DE}{BE}$, so $\frac{AD}{CE} = \frac{\frac{AB \times \cancel{DE}}{BE}}{\frac{BC \times \cancel{DE}}{BD}} = \frac{AB \times BD}{BE \times BC}$ ✓

$\boxed{\frac{5}{3}}$

185)



Since isosceles $BX = XC = 4$ and ABX and ACX are 3-4-5 triangles:

$\text{Area}(ABC) = \frac{1}{2} \times 8 \times 3 = 12$

$\text{Area}(ABC) = \frac{1}{2}(8r) + \frac{1}{2}(5r) + \frac{1}{2}(5r) = 9r$, $9r = 12 \rightarrow r = \frac{4}{3}$

$AI = AX - XI = 3 - r = 3 - \frac{4}{3} = \boxed{\frac{5}{3}}$