<u>OMC Practice Questions for AMC Prep (9/23/2022): Solutions</u>

Set #1 D = RT

1) Jeslyn runs a five mile race. She completes the first two miles running at an average rate of 4 miles per hour. It takes her 48 minutes to complete the last three miles of the race. What was her average speed for the whole five mile race, in miles per hour? Please leave your answer as a reduced fraction (in the form p/q where both p and q are positive integers that don't share a common factor.)

Solution

If she runs 2 miles at a rate of 4 miles per hour, then, the time she took was $t = \frac{2miles}{4mph} = \frac{1}{2}hr.$

Converting 48 minutes to hours, we have $48min = \frac{48}{60}hr = \frac{4}{5}hr.$
Thus, the whole 5 mile run took $\frac{1}{2} + \frac{4}{5} = \frac{5+8}{10} = \frac{13}{10}hrs.$

It follows that her average speed for the whole race was $r = \frac{D}{t} = \frac{5miles}{13/_{10}hr} = \mathbf{\frac{50}{13}}\boldsymbol{mph}$

2) An shuttle bus is making a 200 mile trip. For the first portion of the trip, the bus averages 60 miles an hour. Unfortunately, a rain storm hits and for the rest of the trip (second portion), the bus averages 48 miles per hour. The average speed of the whole 200 mile trip was 50 miles an hour. How long, **<u>in miles</u>**, was the first portion of the trip?

Solution

Let x be the distance of the first portion of the trip. Thus, 200-x is the distance of the second portion of the trip. The time the first portion of the trip takes is $\frac{x}{60}$ hrs and the time the second portion of the trip takes is $\frac{200-x}{48}$ hrs. Since the whole trip of 200 miles had an average speed of 50 miles per hour, the whole trip took $\frac{200}{50} = 4$ hours. This leads to the equation:

$$\frac{x}{60} + \frac{200 - x}{48} = 4$$

Note that 240 is the least common multiple of 60 and 48, so we have:

$$\frac{4x}{240} + \frac{5(200 - x)}{240} = 4$$

$$4x + 1000 - 5x = 4(240)$$
$$-x = 960 - 1000$$
$$-x = -40$$
$$x = \mathbf{40}$$

3) The current in a river is flowing steadily at 3 miles per hour. A motor boat which travels at a constant rate in still water goes downstream 4 miles and then returns to its starting point. The trip takes one hour, excluding the time spent in turning the board around. What is the ratio of the downstream to the upstream rate?

Solution

Let r be the rate of speed (in mph) of the boat in still water. Then the time it took the boat to go downstream is $\frac{4}{r+3}$ hours and the time it took the boat to come back upstream is $\frac{4}{r-3}$ hours. This total time is one hour so we have:

$$\frac{4}{r+3}+\frac{4}{r-3}=1$$

$$\frac{4(r-3)+r(r+3)}{(r-3)(r+3)}=1$$

$$8r=r^2-9$$

$$0=r^2-8r-9$$

$$(r-9)(r+1)=0$$

Since r is positive, it follows that r = 9, the boat traveled at 12 mph going downstream and 6 mph coming back up stream, so the desired ratio is **2:1.**

Set #2 Probability

1) There are 20 marbles in a bag. 8 of the marbles are blue. Megan randomly selects 2 of the marbles at one time. What is the probability both are blue?

Solution

We have 20 choices for the first marble and 19 for the second. We can choose a blue marble as the first marble in 8 ways, and 7 ways for the second marble. It follows that the corresponding probability is $\frac{8}{20} \times \frac{7}{19} = \frac{2}{5} \times \frac{7}{19} = \mathbf{\frac{14}{95}}$.

2) Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

Solution

Let Mel's chance of winning be 2x, which means Chelsea's chance of winning is x. Since Alex wins half the time, we have $2x + x = \frac{1}{2}$. It follows that $x = \frac{1}{6}$. Thus, Chelsea's chance of winning is $\frac{1}{6}$, and Mel's chance of winning is $\frac{1}{3}$. The probability of Alex winning three times, followed by Mel winning twice, followed by Chelsea winning is $(\frac{1}{2})^3(\frac{1}{3})^2\frac{1}{6}$. Of course, these number of wins can happen in several ways. In particular, they can happen in the number of ways we can permute 3 A's 2 M's and 1 C. This is equal to $\frac{6!}{3!2!1!}$. It follows that the desired probability is the product of these two items:

$$\frac{6!}{3!\,2!\,1!} \times \frac{1}{8} \times \frac{1}{9} \times \frac{1}{6} = \frac{2 \times 5 \times 6}{8 \times 9 \times 6} = \mathbf{\frac{5}{36}}$$

3) Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?

Solution

There are $2^8 = 256$ possible ways for the coin flips to occur, each equally likely. We must count all the ways where no two adjacent people flip heads. There must be either 0, 1, 2, 3 or 4 people who flip heads.

There is 1 way that 0 people flip heads.

There are 2 ways that 4 people flip heads (either the odd numbered people all get heads or the even numbered people).

There are 8 ways for 1 person to flip heads.

For two people to flip heads, we have $\binom{8}{2} - 8 = 20$ ways to do it because there are $\binom{8}{2}$ ways total to choose 2 people out of 8, but of those possibilities, 8 are two adjacent pairs.

The trickiest count is for 3 people flipping heads. Let's split our counting into 2 groups, all the combinations where person 1 flips heads and all the combinations where person 1 flips tails. If person 1 flips heads, then person 2 and person 8 can not. This leaves 2 people out of persons 3, 4, 5, 6 and 7 to flip heads. There are 6 ways to do this ({3,5}, {3, 6}, {3, 7}, {4, 6}, {4, 7}, {5, 7} or we can use our old reasoning of $\binom{5}{2} - 4 = 6$.). If person 1 doesn't flip heads, then we have these cases: (a) both persons 8 and 2 flip heads – this can be done in 3 ways, (b) one of person 8 and 2 flips heads – this can be done in 6 ways, and (c) neither of person 8 nor person 2 flip heads – this can be done in 1 way. This is a total of 14 ways.

Thus there are 1 + 2 + 8 + 20 + 16 = 47 ways. It follows that the desired probability is $\frac{\mathbf{47}}{\mathbf{256}}$.

Set #3 Logs

1) For how many integral values of x can a triangle of positive area be formed having side lengths of $\log_2 x$, $\log_4 x$, and 3?

Solution

Note that 1, 2 and 3 "just barely" don't make a triangle. If x = 4, then the three values are 1, 2 and 3. But, if we try x = 5, then the smaller sides of $\log_2 5$ and $\log_4 5$ are both greater than 1 and 2, respectively, so they would form a valid triangle. We can plug in successive values of x and they will continue working until $3 + \log_4 x \leq \log_2 x$. Recall that $log_2 x = \frac{log_4 x}{log_4 2} = 2log_4 x$. Thus, if we let $y = log_4 x$, then $log_2 x = 2y$. Thus, we want to know for which value of y, $3 + y \leq 2y$. This is when $y \geq 3$, which corresponds to when $log_4 x \geq 3$. This is when $x \geq 64$. It follows that the three values make valid lengths of a triangle for all x such that $4 < x < 64$. There are 63 – 4 = **59** such integral values of x.

2) What is the value of the following summation: $log_{n!}1 + log_{n!}2 + log_{n!}3 + \cdots log_{n!}n$?

Solution

$$\sum_{i=1}^{n} log_{n!} i = log_{n!}\left(\prod_{i=1}^{n} i\right) = log_{n!} n! = \mathbf{1}$$

Note: The sigma means to add the terms log 1 + log 2 + log 3 + … + log n, where the base is n!. The log addition rule says that to put these in a single log, we multiply each item we are taking the log of. The pi sign means exactly that, to multiply each term you get when you plug i into the expression on the inside and the product of each integer from 1 to n is defined as n factorial (n!).

3) Positive real numbers $x \neq 1$, $y \neq 1$, satisfy $log_2 x = log_y 16$ and $xy = 64$. What is $(log_2 \frac{x}{y})^2$?

Take log base 2 of both sides of the second equation to yield $log_2 x + log_2 y = 6$. Solve for $\log_2 x$:
$log_2 x = 6 - log_2 y$

Substitute this into the first equation:

$6 - log_2 y = log_y 16$
$6 - log_2 y = log_y 2^4$
$6 - log_2 y = 4log_y 2$

Let $A = \log_2 y$, and observe that the right hand side is 1/A and substitute:

$6 - A = \frac{4}{A}$
$6A - A^2 = 4$
$A^2 - 6A + 4 = 0$
$A = 3 \pm \sqrt{5}$, after applying the quadratic formula.
It follows that $\log_2 y$ is one of these roots and $\log_2 x$ is the other.

Recall that we want to solve for: $(log_2 \frac{x}{y})^2 = (log_2 x - log_2 y)^2$. Although we don't know which value equals $\log_2 x$ and $\log_2 y$, if we plug in both possibilities, we see that the difference between the logs is $\pm 2\sqrt{5}$. When we square either value, we get **20.**

Set #4 Counting

1) How many permutations are there of the letters in the word CATTRAP? Simplify your answer to a single integer.

There are 7 letters, and 2 letters appear twice.

Using the formula for permutations, we have $\frac{7!}{2!2!} = \frac{5040}{4} = \mathbf{1260}$

2) At a twins and triplets convention, there were 9 sets of twins and 6 sets of triplets, all from different families. Each twin shook hands with all the twins except his/her sibling and with half the triplets. Each triplet shook hands with all the triplets except his/her siblings and with half the twins. How many handshakes took place?

For each individual, count how many handshakes they participated in. Sum this total up, and divide by 2. (This is an observation based on the handshaking lemma, that, for each handshake, 2 people participate in it.) For each of the 18 twins, they shook hands with 16 other twins and 9 triplets. Each of the 18 triplets shook hands with 15 other triplets and 9 twins. Adding this up we get:

18 x (16 + 9) + 18 x (15 + 9) = 18 x 49

Finally, we divide this by 2, to get 9 x 49 = **441** total handshakes.

3) How many ordered quadruplets (a_1, a_2, a_3, a_4) of non-negative integers, where at least one of the integers is even, satisfy the equation $a_1 + a_2 + a_3 + a_4 = 100$? Please express your answer in the form $\binom{w}{x} - \binom{y}{z}$. (Note that the values of w, x, y and z will be integers, but not necessarily all distinct.)

Let's count all of the solutions (in non-negative integers) first, without worrying about the parity of those solutions.

We can think of the answer to the question as an arrangement of 100 stars, representing the 100 on the right hand side, along with 4 – 1 = 3 dividing bars. Here is a sample:

*****…***|**|***…****|***
40 2 55 3

For example, we might have 40 stars, followed by a bar, then 2 stars, followed by a bar, then 55 stars, followed by a bar, ending with 3 stars. Notice that each unique arrangement of stars and bars corresponds to a different solution for (a_1, a_2, a_3, a_4) and each possible solution corresponds to a unique arrangement of stars and bars. (This is called a one to one correspondence.) It follows that the answer to our original subquestion, the number of total solutions in non-negative integers to that equation is equal to the number of ways to arrange 100 stars and 3 bars. We can solve this via the permutation formula or combination formula. (We are permuting 103 objects, of which there are 100 copies of object 1 and 3 copies of object 2. Or, we are choosing 3 slots out of 103 slots to place the bars. Or, we are choosing 100 slots out of 103 to place the stars.) One way to express the total number of solutions is $\binom{103}{3}$.

But, this is counting a few items we don't want to count. Namely, if ALL the integers are odd, we don't want to count that solution. So, let's subtract those out.

Because a_1, a_2, a_3, a_4, are odd, there exist non-negative integers x, y, z and w such that $a_1 = 2x + 1$, $a_2 = 2y + 1$, $a_3 = 2z + 1$, $a_4 = 2w + 1$. Substitute to get:

$2x + 1 + 2y + 1 + 2z + 1 + 2w + 1 = 100$
$2(x + y + z + w) = 96$
$x + y + z + w = 48$

This is the same exact stars and bars problem! This equation has $\binom{51}{3}$ solutions.

It follows that the final answer to the question is $\boldsymbol{\binom{103}{3} - \binom{51}{3}}$**.** (Note: Both combinations can be represented as a different combination, so there are four correct answers.)

Set #5 Divisors, GCD, LCM

1) How many positive integers less than 500 have an odd number of divisors?

Only perfect squares have an odd number of divisors. $22^2 = 484$ and $23^2 = 529$. It follow sthat the answer to the question is **22.**

2) Let M be the least common multiple of all the integers 10 through 30, inclusive. Let N be the least common multiple of M, 32, 33, 34, 35, 36, 37, 38, 39 and 40. What is the value of $\frac{N}{M}$?

The value we desire is $\frac{lcm(10,11,12,\ldots,30,32,33,\ldots,40)}{lcm(10,11,12,\ldots,30)}$. All of the prime factors from below are present above, so these will cancel. What we are looking for are unique leftover prime factors created by the new terms.

Note that 2^4 is the highest power of 2 that appears in $lcm(10,11,12,\ldots,30)$, but that $32 = 2^5$. This means that the numerator has an extra factor of 2 that the denominator doesn't have.

33 = 3 x 11 and both terms are already in the denominator
34 = 2 x 17 and both terms are already in the denominator
35 = 5 x 7 and both terms are already in the denominator
36 = 2^2 x 3^2 and both terms are already in the denominator
37 is a prime, so this factor is in the numerator and not the denominator
38 = 2 x 19 and both terms are already in the denominator
39 = 3 x 13 and both terms are already in the denominator
40 = 2^3 x 5 and both terms are already in the denominator

It follows that the fraction equals 2 x 37 = **74.**

3) A positive integer n has 60 divisors and 7n has 80 divisors. What is the greatest integer k such that 7^k divides n?

The formula for # of divisors of an integer is $(a_1 + 1)(a_2 + 1)\ldots(a_k + 1)$ where the a_i's are the exponents to each prime in the prime factorization of the integer. Without loss of generality, let a_1 equal the exponent for 7 and $a_1, \ldots, a_k$ be the corresponding exponents in the prime factorization for n. It follows that:

$(a_1 + 1)(a_2 + 1)\ldots(a_k + 1) = 60$
$(a_1 + 2)(a_2 + 1)\ldots(a_k + 1) = 80$

Divide these two equations to get $\frac{a_1+1}{a_1+2} = \frac{3}{4}$. The only solution to this equation is $a_1 = 2$. It follows that k = **2.**