Engineering Analysis ENG 3420 Fall 2009

Dan C. Marinescu

Office: HEC 439 B

Office hours: Tu-Th 11:00-12:00

Lecture 21

- Last time:
  - Relaxation
  - Non-linear systems
  - Random variables, probability distributions, Matlab support for random variables
- Today
  - Histograms
  - Linear regression
  - Linear least squares regression
  - Non-linear data models
- Next Time
  - Multiple linear regression
  - General linear squares

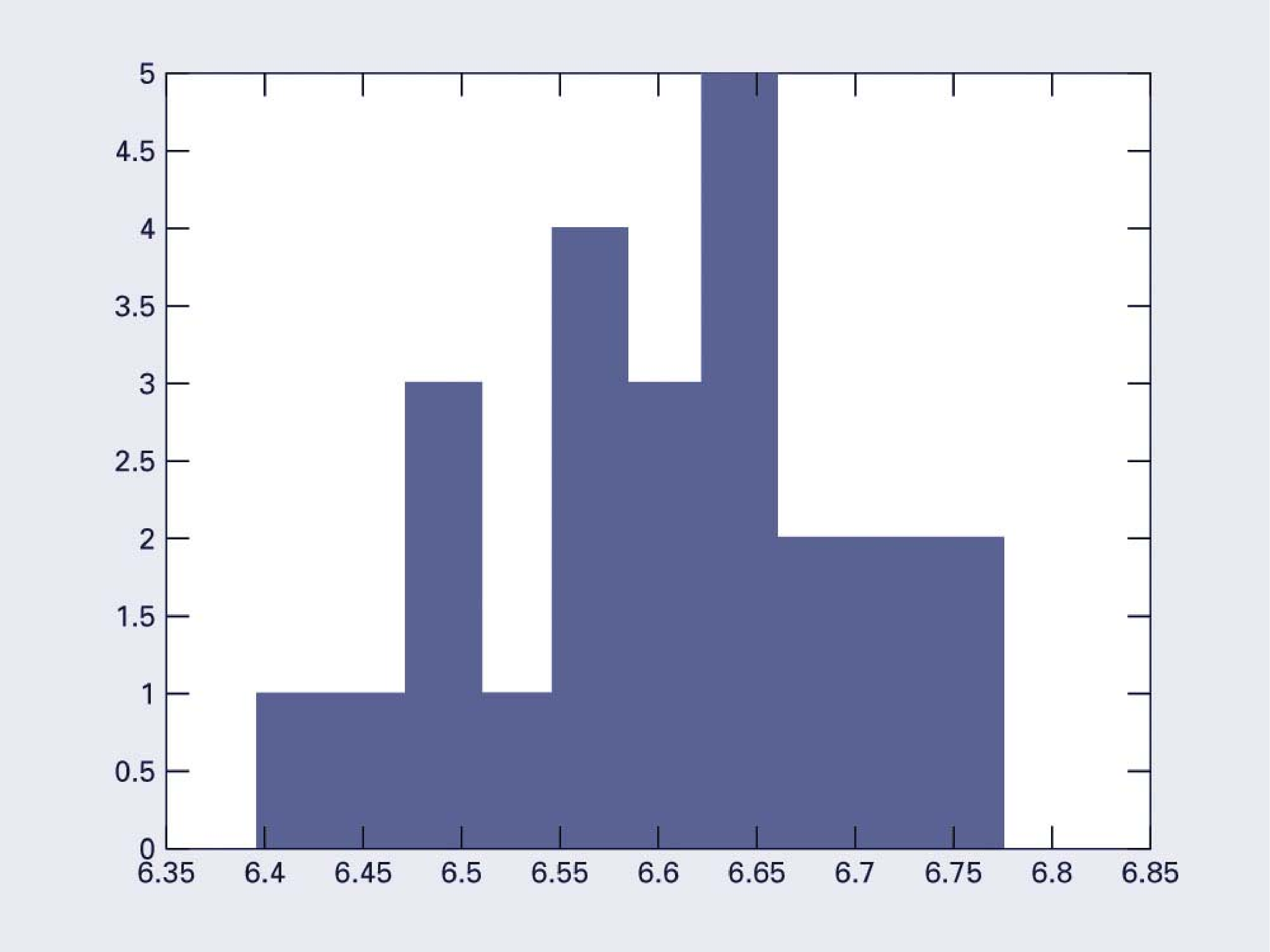
Statistics built-in functions

- Built-in statistics functions for a column vector *s*:
  - mean(*s*), median(*s*), mode(*s*)
    - Calculate the mean, median, and mode of *s*. `mode` is a part of the statistics toolbox.
  - min(*s*), max(*s*)
    - Calculate the minimum and maximum value in *s*.
  - var(*s*), std(*s*)
    - Calculate the variance and standard deviation of *s*
- If a matrix is given, the statistics will be returned for each column.

Histograms

- [*n*, *x*] = hist(*s*, *x*)
  - Determine the number of elements in each bin of data in *s*.
  - *x* is a vector containing the center values of the bins.
- [*n*, *x*] = hist(*s*, *m*)
  - Determine the number of elements in each bin of data in *s* using *m* bins.
  - *x* will contain the centers of the bins.
  - The default case is *m*=10
- hist(*s*, *x*) or hist(*s*, *m*) or hist(*s*)
  - With no output arguments, hist will actually produce a histogram.

Histogram Example



Linear Least-Squares Regression

- Linear least-squares regression is a method to determine the “best” coefficients in a linear model for given data set.
- “Best” for least-squares regression means minimizing the sum of the squares of the *estimate* residuals. For a straight line model, this gives:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i\right)^2$$

- This method will yield a unique line for a given set of data.

Least-Squares Fit of a Straight Line

- Using the model:

$$y = a_0 + a_1 x$$

the slope and intercept producing the best fit can be found using:

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

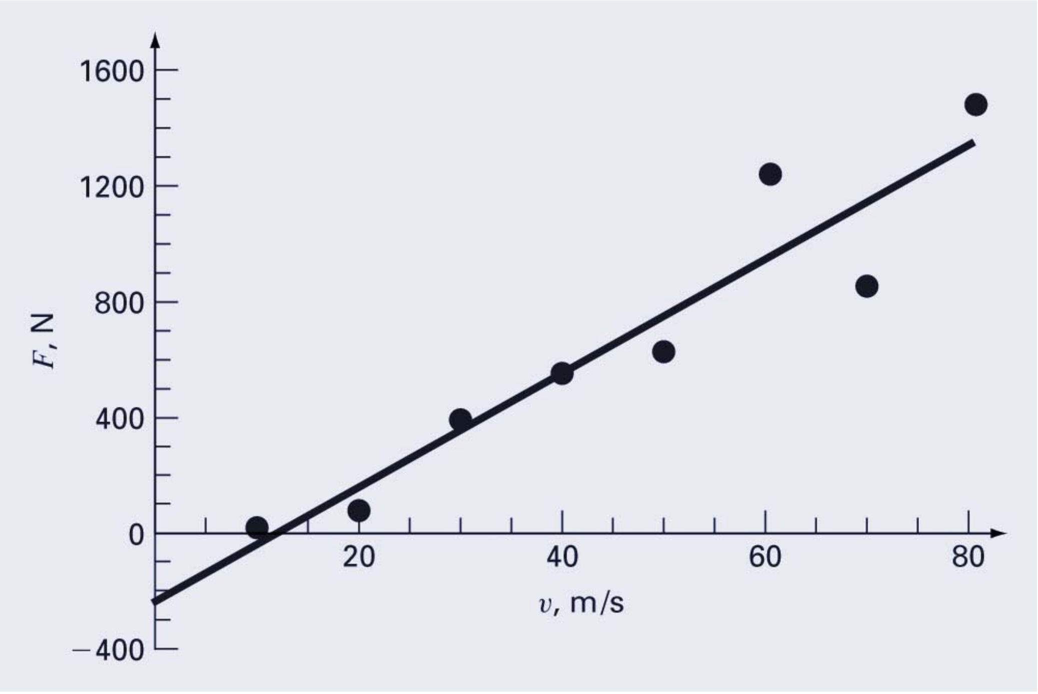
Example

	V (m/s)	F (N)		
i	x_i	y_i	$(x_i)^2$	x_iy_i
1	10	25	100	250
2	20	70	400	1400
3	30	380	900	11400
4	40	550	1600	22000
5	50	610	2500	30500
6	60	1220	3600	73200
7	70	830	4900	58100
8	80	1450	6400	116000
Σ	360	5135	20400	312850

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{8(312850)-(360)(5135)}{8(20400)-(360)^2} = 19.47024$$

$$a_0 = \bar{y} - a_1\bar{x} = 641.875 - 19.47024(45) = -234.2857$$

$$F_{est} = -234.2857 + 19.47024v$$



Nonlinear models

- Linear regression is predicated on the fact that the relationship between the dependent and independent variables is linear - this is not always the case.
- Three common examples are:

$$\text{exponential}: \qquad y = \alpha_1 e^{\beta_1 x}$$

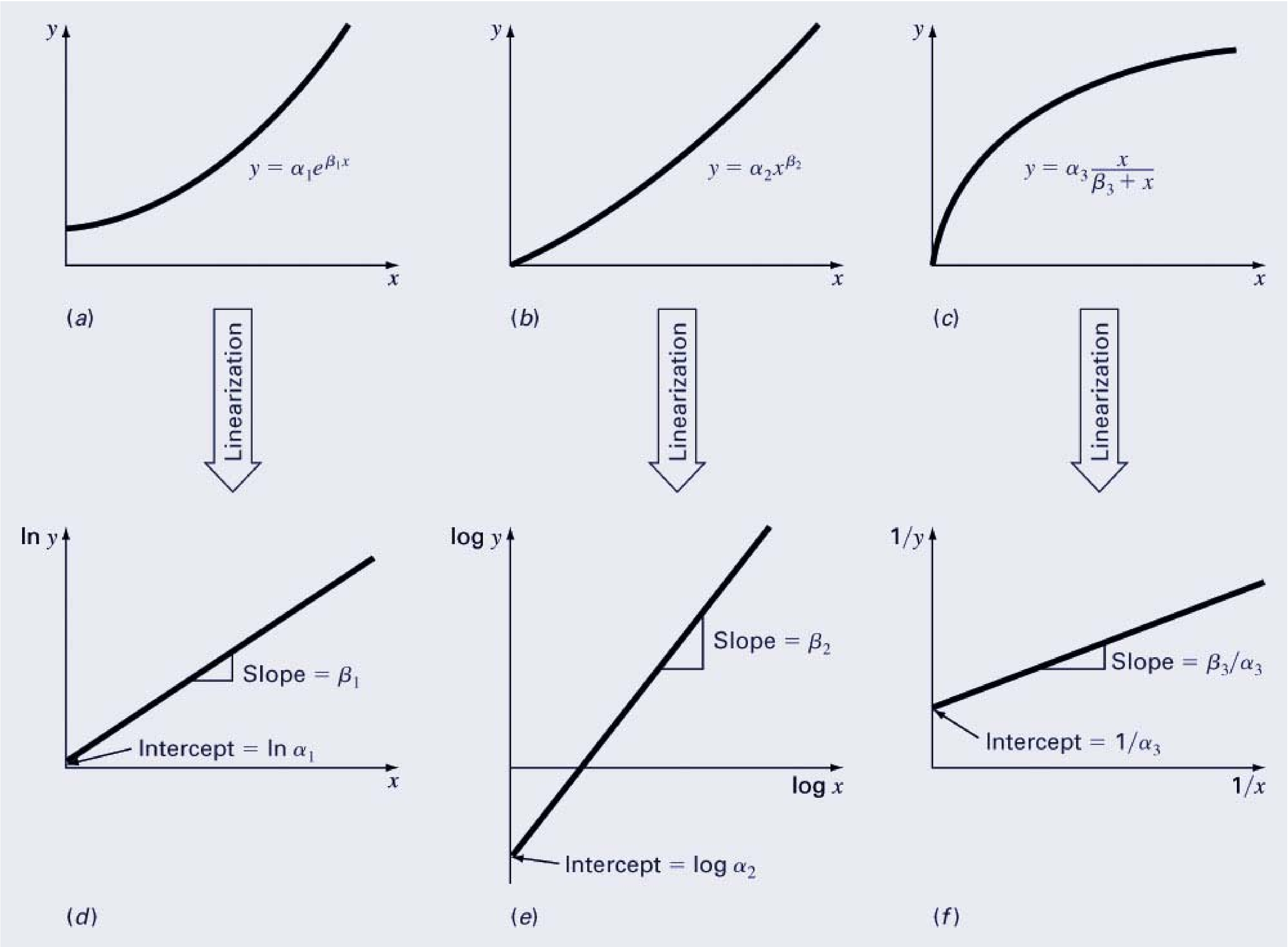
$$\text{power}: \qquad y = \alpha_2 x^{\beta_2}$$

$$\text{saturation-growth-rate}: \qquad y = \alpha_3 \frac{x}{\beta_3 + x}$$

Linearization of nonlinear models

Model	Nonlinear	Linearized
exponential:	$y = \alpha_1 e^{\beta_1 x}$	$\ln y = \ln \alpha_1 + \beta_1 x$
power:	$y = \alpha_2 x^{\beta_2}$	$\log y = \log \alpha_2 + \beta_2 \log x$
saturation-growth-rate:	$y = \alpha_3 \dfrac{x}{\beta_3 + x}$	$\dfrac{1}{y} = \dfrac{1}{\alpha_3} + \dfrac{\beta_3}{\alpha_3} \dfrac{1}{x}$

Transformation Examples



Linear Regression Program

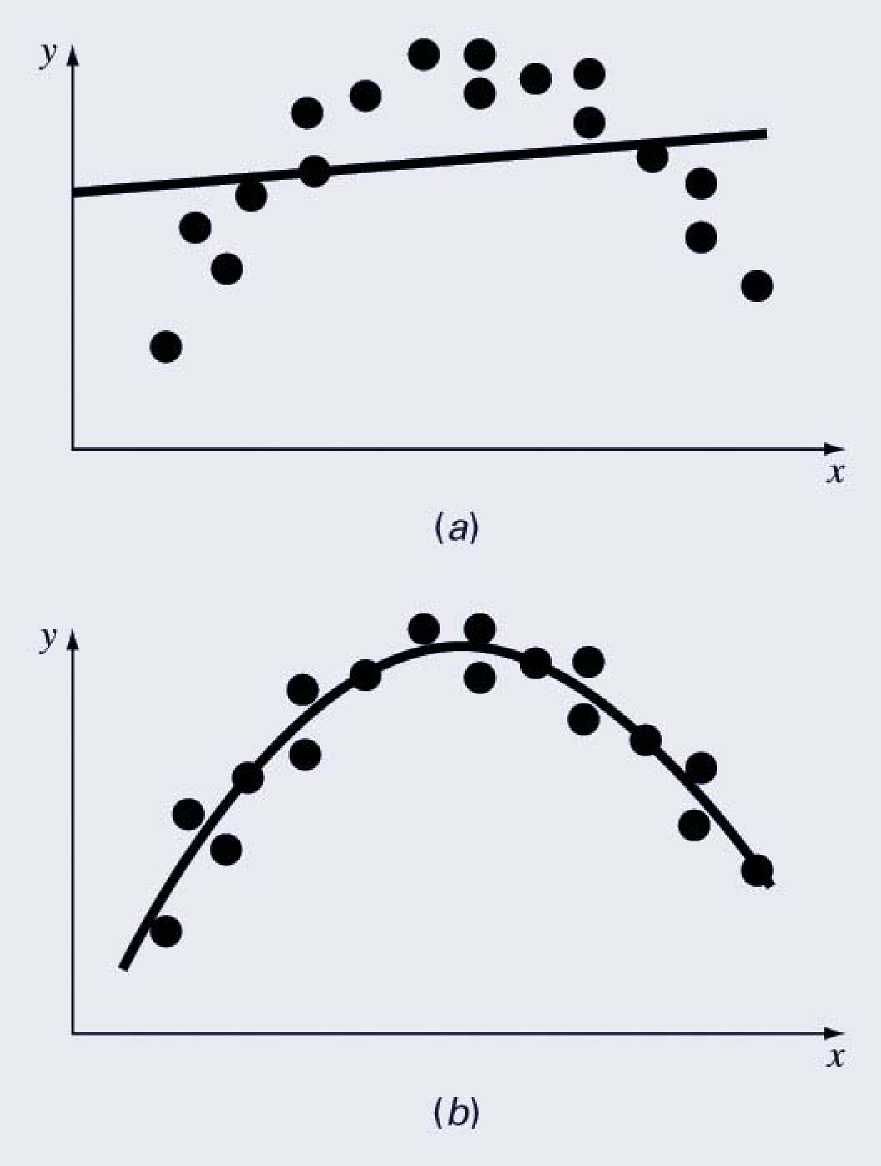
```
function [a, r2] = linregr(x,y)
% linregr: linear regression curve fitting
%   [a, r2] = linregr(x,y): Least squares fit of straight
%               line to data by solving the normal equations

% input:
%   x = independent variable
%   y = dependent variable
% output:
%   a = vector of slope, a(1), and intercept, a(2)
%   r2 = coefficient of determination

n = length(x);
if length(y)~=n, error('x and y must be same length'); end
x = x(:); y = y(:);     % convert to column vectors
sx = sum(x); sy = sum(y);
sx2 = sum(x.*x); sxy = sum(x.*y); sy2 = sum(y.*y);
a(1) = (n*sxy-sx*sy)/(n*sx2-sx^2);
a(2) = sy/n-a(1)*sx/n;
r2 = ((n*sxy-sx*sy)/sqrt(n*sx2-sx^2)/sqrt(n*sy2-sy^2))^2;
% create plot of data and best fit line
xp = linspace(min(x),max(x),2);
yp = a(1)*xp+a(2);
plot(x,y,'o',xp,yp)
grid on
```

Polynomial least-fit squares

- MATLAB has a built-in function polyfit that fits a least-squares n-th order polynomial to data:
  - p = polyfit(x, y, n)
    - x: independent data
    - y: dependent data
    - n: order of polynomial to fit
    - p: coefficients of polynomial $f(x)=p_1x^n+p_2x^{n-1}+\ldots+p_nx+p_{n+1}$
- MATLAB's polyval command can be used to compute a value using the coefficients.
  - y = polyval(p, x)

Polynomial Regression

- The least-squares procedure from can be extended to fit data to a higher-order polynomial. The idea is to minimize the sum of the squares of the estimate residuals.
- The figure shows the same data fit with:
  a) A first order polynomial
  b) A second order polynomial



Process and Measures of Fit

- For a second order polynomial, the best fit would mean minimizing:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 \right)^2$$

- In general, this would mean minimizing:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - a_2 x_i^2 - \cdots - a_m x_i^m \right)^2$$

- The standard error for fitting an m^{th} order polynomial to n data points is:

$$s_{y/x} = \sqrt{\frac{S_r}{n - (m+1)}}$$

- because the m^{th} order polynomial has (m+1) coefficients.
- The coefficient of determination r^2 is still found using:

$$r^2 = \frac{S_t - S_r}{S_t}$$