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Linear Algebra Concepts

Vector Spaces To define the concept of a vector space we first need to introduce two basic algebraic structures, the group and the field. A *group* is a set G with one binary operation "$\cdot$", called multiplication, which satisfies three conditions

1. Associative law: $\forall (a, b, c) \in G \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

2. Identity element: There is an identity element $e \in G$ such that $a \cdot e = e \cdot a = a, \quad \forall a \in G$.

3. Inverse element: $\forall a \in G, \exists a^{-1}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

A group G whose operation satisfies the commutative law (i.e., $a \cdot b = b \cdot a$) is a *commutative*, or *Abelian*, group.

A *field* is a set F equipped with two binary operations, addition and multiplication, with the following properties:

1. under addition, F is an Abelian group with the identity (or neutral) element 0 such that $0 + a = a, \quad \forall a \in F$.

2. under multiplication, the nonzero elements form an Abelian group with neutral element 1 such that $1 \cdot a = a, \quad \forall a \in F$, and $0 \cdot a = 0, \quad \forall a \in F$. The additive and multiplicative identity elements are different, $0 \neq 1$.

3. the distributive law holds: $a \cdot (b + c) = a \cdot b + a \cdot c$.

A *vector space* $\mathcal{A}$ assumes three objects:

1. An Abelian group $(V, +)$ whose elements are called "vectors" and whose binary operation "+" is called *addition*,

2. A field F (usually $\mathbb{R}$, the real numbers, or $\mathbb{C}$, the complex numbers), whose elements are called "scalars", and

3. An operation called "multiplication with scalars" and denoted by "$\cdot$", which associates to any scalar $c \in F$ and vector $\alpha \in V$ a new vector $c \cdot \alpha \in V$. The multiplication by scalars operation has the following properties

$$c \cdot (\alpha + \beta) = c \cdot \alpha + c \cdot \beta$$

$$(c + c') \cdot \alpha = c \cdot \alpha + c' \cdot \alpha$$

$$(c \cdot c') \cdot \alpha = c \cdot (c' \cdot \alpha), \quad 1 \cdot \alpha = \alpha$$

where $\alpha, \beta \in V$ and $c, c' \in F$.

Observations:

(a) Often we omit the "·" symbol and write the product of two scalars as cc' instead of $c \cdot c'$ and the product of a scalar with a vector as $c\alpha$ instead of $c \cdot \alpha$.

(b) Given n scalars $\{c_1, c_2, \ldots, c_n\} \in \mathbb{R}$, then the set of n vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathbb{R}^n$ are *linearly independent* if

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = 0 \quad \Longrightarrow \quad c_1 = c_2 = \ldots = c_n = 0.$$

Vectors that are not linearly independent are called *linearly dependent.*

A *subspace* $\mathcal{S}$ of a vector space $\mathcal{A}$ is a subset of $\mathcal{A}$ which is closed with respect to the operations of addition and scalar multiplication. This means that the sum of two vectors in $\mathcal{S}$ is in $\mathcal{S}$. For any vector and any scalar, the product of a vector with a scalar is also a vector, $\forall \alpha \in \mathcal{S}$ and $\forall c \in \mathbb{R}$ then $c\alpha \in \mathcal{S}$.

Examples of subspaces:

1. The set of polynomials of degree at most m is a subspace of the vector space of all polynomials which is a subspace in the vector space of complex valued continuous functions $C_R(\mathbb{R})$.

2. The set of all continuous functions $f(x)$ defined for $0 \leq x \leq 2$ is a subspace of the linear space of all functions defined on the same domain.

The set of all linear combinations of any set of vectors of a vector space $\mathcal{A}$ is a subspace of $\mathcal{A}$. Given $c', c'_1, c'_2, \ldots c'_n \in F$ the following two identities allow us to prove this statement

$$(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_m\alpha_m) + (c'_1\alpha_1 + c'_2\alpha_2 + \ldots + c'_m\alpha_m) = \\ (c_1 + c'_1)\alpha_1 + (c_2 + c'_2)\alpha_2 + \ldots + (c_m + c'_m)\alpha_m$$

$$c'(c_1\alpha_1 + c_2\alpha_2 + \ldots c_m\alpha_m) = (c'c_1)\alpha_1 + (c'c_2)\alpha_2 + \ldots (c'c_m)\alpha_m$$

The subspace consisting of all linear combinations of a set of vectors of $\mathcal{A}$ is the smallest subset[1] containing all the given vectors. The set of vectors *spans* the subspace.

A linearly independent subset of vectors which spans the whole space is called a *basis of a vector space* . A vector space $\mathcal{A}$ is *finite dimensional* if and only if it has a finite basis.

If $\{b_1, b_2, \ldots, b_m\}$ and $\{c_1, c_2, \ldots, c_n\}$ are two bases for a vector space and m and n are finite then $m = n$.

The minimum number of vectors in any basis of a finite-dimensional vector space $\mathcal{A}$ is called the *dimension of a vector space*, $\dim(\mathcal{A})$. For example, the ordinary space $\mathbb{R}^3$ can be spanned by three vectors, $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$.

<u>n-Dimensional Real Euclidean Vector Space</u> Consider now an n-dimensional real vector space, $\mathbb{R}^n$. If for every pair of vectors $\alpha, \beta \in \mathbb{R}^n$ we have an associated real number (α, β) such that the following four conditions are satisfied

1. $(\alpha, \beta) = (\beta, \alpha)$
2. $(c\alpha, \beta) = c(\alpha, \beta)$, if $c \in \mathbb{R}$
3. $(\alpha + \gamma, \beta) = (\alpha, \beta) + (\gamma, \beta), \quad \forall \gamma \in \mathbb{R}^n$
4. $(\alpha, \alpha) \geq 0$ and $(\alpha, \alpha) = 0$ if and only if $\alpha = 0$

[1] The smallest subset is the subset with the smallest cardinality.

then we say that we have an n-dimensional *Euclidean space* and that (α, β) is the *inner product* of vectors α and β. All the facts known from Euclidean geometry can be established in an Euclidean space. The *length of a vector* α in a Euclidean space is defined to be the real number

$$\mid \alpha \mid \;\; = \;\; \sqrt{(\alpha, \alpha)}.$$

The *angle between two vectors* α and β is the real number θ:

$$\theta = \arccos \frac{(\alpha, \beta)}{\mid \alpha \mid \;\; \mid \beta \mid} \qquad \Longrightarrow \qquad \cos \theta = \frac{(\alpha, \beta)}{\mid \alpha \mid \;\; \mid \beta \mid}.$$

If $(\alpha, \beta) = 0$, where $\alpha \neq 0$ and $\beta \neq 0$, then $\theta = \pi/2$, and $\cos \theta = 0$. Two vectors α and β in a Euclidean space are *orthogonal* if

$$(\alpha, \beta) = 0.$$

A set of n vectors $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ is an *orthogonal basis* in an n-dimensional Euclidean space if the vectors in the set are pairwise orthogonal. If, in addition, each vector has a unit length, then the vectors form an *orthonormal basis* and satisfy the condition

$$(e_i, e_j) = \delta_{i,j} = \begin{cases} 0 & \text{if } \; i \neq j \\ 1 & \text{if } \; i = j \end{cases} \qquad 1 \leq (i, j) \leq n.$$

with $\delta_{i,j}$ the Kronecker delta function.

It is relatively easy to show that every n-dimensional Euclidean space contains orthogonal bases. Given an orthonormal basis $\{e_1, e_2, \ldots e_n\}$ of an n-dimensional Euclidean space we can express any two vectors as

$$\alpha = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n$$

and

$$\beta = b_1 e_1 + b_2 e_2 + \ldots + b_n e_n.$$

Then, we use the fact that $(e_i, e_j) = \delta_{i,j}$ to show that

$$(\alpha, e_i) = a_i$$

and that

$$(\alpha, \beta) = \sum_{i=1}^{n} a_i b_i.$$

The simplest functions defined on vector spaces are the linear transformations. The function $A(\alpha)$ is a *linear form (function)* if

$$A(\alpha; \beta) = A(\alpha) + A(\beta)$$

and

$$A(c\alpha) = cA(\alpha).$$

Consider two vector spaces over the same field F, called $\mathcal{A}$ and $\mathcal{B}$. Let $\alpha \in \mathcal{A}$, $\beta \in \mathcal{A}$, $c \in F$, and $A(\alpha) \in \mathcal{B}$. We are given a function A which maps vectors in $\mathcal{A}$ to vectors in $\mathcal{B}$.

The function $A(\alpha;\beta)$ is said to be a *bilinear form (function)* of vectors α and β if

1. For any fixed β, $A(\alpha;\beta)$ is a linear function of α,

2. For any fixed α, $A(\alpha;\beta)$ is a linear function of β.

This implies that if $\mathcal{A}$ and $\mathcal{B}$ are two vector spaces over the same field F and we consider variable vectors $\alpha, \alpha' \in \mathcal{A}$ and $\beta, \beta' \in \mathcal{B}$ and scalars $a, b, c, d \in F$ the function $A(\alpha, \beta)$ with values in F is a *bilinear function* if

$$A(a\alpha \ + \ b\alpha'; \beta) \ = \ aA(\alpha;\beta) \ + \ bA(\alpha';\beta)$$

and

$$A(\alpha; c\beta \ + \ d\beta') \ = \ cA(\alpha;\beta) \ + \ dA(\alpha;\beta')$$

A bilinear function is *symmetric* if

$$A(\alpha;\beta) = A(\beta;\alpha).$$

The inner product of two vectors in an Euclidean vector space is an example of a symmetric bilinear function.

If $A(\alpha;\beta)$ is a symmetric quadratic form, then the function $A(\alpha;\alpha)$ is called a *quadratic form*. A quadratic form $A(\alpha;\alpha)$ is *positive definite* if for every vector $\alpha \neq 0$

$$A(\alpha;\alpha) > 0.$$

The bilinear form $A(\alpha;\beta)$ is called the *polar form associated* with the quadratic form $A(\alpha;\alpha)$. It can be shown that $A(\alpha;\beta)$ is uniquely determined by its quadratic form.

Now we can provide an alternative definition of an Euclidean vector space as *a vector space* with a positive definite quadratic form $A(\alpha;\alpha)$. The inner product (α, β) of two vectors is the value of the bilinear form $A(\alpha;\beta)$ associated with $A(\alpha;\alpha)$.

Linear Operators and Matrices A rectangular array of elements of a field F with m rows and n columns, A, is called a *matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Matrix A can be interpreted as a linear map from the vector space of dimension n, F^n, to the vector space of dimension m, F^m equipped with the canonical base. If $n = m$, then A is a linear map from F^n to itself.

Let $\alpha_1 = (a_{11}\ a_{12}\ \dots\ a_{1n})$, $\alpha_2 = (a_{21}\ a_{22}\ \dots\ a_{2n})$, $\dots$ $\alpha_m = (a_{m1}\ a_{m2}\ \dots\ a_{mn})$ be a set of vectors spanning a subspace of dimension m of a vector space $V_n(F)$ called the *row space* of the $m \times n$ matrix A.

The elementary row operations on a matrix are:
1. the interchange of any two rows,

2. multiplication of all the elements of a row by a constant $c \in F$, and
3. addition of any multiple of a row to any other row.

Two matrices are row-equivalent if one is obtained from the other by a finite sequence of row operations.

In the case $n = m$ the *identity matrix* $I = [a_{ij}]$ is the matrix with $a_{ii} = 1$ and $a_{ij} = 0$, if $i \neq j$. For example, the identity matrix in a vector space of dimension 8 is

$$I_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A *permutation matrix* P is an identity matrix with rows and columns permuted.

The *determinant* of an $n \times n$ matrix $A = [a_{ij}]$ is the polynomial

$$\det(A) = \mid A \mid = \sum_{\phi} \operatorname{sgn}(\phi)\ a(1, 1\phi)\ a(2, 2\phi) \ldots a(n, n\phi).$$

ϕ denotes one of the $n!$ different permutations of integers $1, 2, \ldots n$. If ϕ is an even permutation then $\operatorname{sgn}(\phi) = +1$. For an odd permutation $\operatorname{sgn}(\phi) = -1$.

The determinant can be written as:

$$\det(A) = A_{i1}a_{i1} + A_{i2}a_{i2} + \ldots A_{in}a_{in}.$$

Here the coefficient A_{ij} of a_{ij} is called the *cofactor* of a_{ij}. A cofactor is a polynomial in the remaining rows of A and can be described as the *partial derivative* $(\partial \mid A \mid / \partial a_{ij})$ of A. The cofactor polynomial contains only entries from an $(n-1) \times (n-1)$ matrix M_{ij} (also called a "minor") obtained from A by eliminating row i and column j.

If we permute two rows of A then the sign of the determinant $\mid A \mid$ changes. The determinant of the transpose of a matrix is equal to the determinant of the original matrix:

$$\mid A^T \mid = \mid A \mid.$$

If two rows of A are identical then:

$$\mid A \mid = 0.$$

A square $(n \times n)$ matrix is *triangular* if all entries below the diagonal are zero. The determinant of a triangular matrix is the product of its diagonal elements. To compute the determinant $\mid A \mid$ one should perform elementary row operations on matrix A and reduce it to a triangular form.

The *characteristic polynomial of matrix A* is:

$$c(\lambda) \equiv \det(A - \lambda I) \qquad \text{or} \qquad c(\lambda) \equiv \mid A - \lambda I \mid.$$

The *trace of matrix* A is the sum of its diagonal elements.

$$\mathrm{Tr}(A) = \sum_i a_{ii}.$$

The *trace of an operator* $\mathbf{A}$ is the trace of A, the matrix representation of $\mathbf{A}$. Given any two matrices A and B over F and a scalar $c \in F$ it is easy to show that the trace has the following properties:

1. It is cyclic

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

2. Linearity

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \qquad\qquad \mathrm{Tr}(cA) = c\mathrm{Tr}(A).$$

3. Invariance under the *similarity transformation*, S

$$\mathrm{Tr}(SAS^\dagger) = \mathrm{Tr}(S^\dagger SA) = \mathrm{Tr}(A).$$

The determinant, the trace, and the characteristic polynomial are quantities associated with any linear map from a finite dimensional vector space into itself.

Now we can express a bilinear form $A(\beta;\gamma)$ in terms of the projections of the two vectors, β and γ, namely $b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_{n-1}$ on the orthonormal basis $e_1, e_2, \ldots, e_n$.

$$A(\beta;\gamma) = (b_1e_1 + b_2e_2 + \ldots + b_ne_n \ ; \ c_1e_1 + c_2e_2 + \ldots + c_ne_n).$$

But A is a bilinear function, thus

$$A(\beta;\gamma) = \sum_{i,j=1}^{n} A(e_i;e_j)b_ic_j.$$

Here $A(e_i;e_j)$ is a real number that we denote it by a_{ij}. Then

$$A(\beta;\gamma) = \sum_{i,j=1}^{n} a_{ij}b_ic_j.$$

The matrix with elements a_{ij}, $A = [a_{ij}], \quad 1 \leq (i,j) \leq n$ is called the *matrix of the bilinear form* $A(\beta;\gamma)$ relative to the basis $e_1, e_2, \ldots e_n$.

Hermitian Operators in a Complex n-Dimensional Euclidean Vector Space It is necessary to consider vector spaces over fields other than $\mathbb{R}$, the field of real numbers. $\mathbb{C}^n$, the n-dimensional vector space over $\mathbb{C}$, the field of complex numbers is of particular interest.

The concepts of linear and bilinear transformations introduced for an n-dimensional Euclidean vector space over the field of real numbers, extend to finite-dimensional vector spaces over other fields. For example, *the inner product* in a vector space $\mathbb{C}^n$ over a field $\mathbb{C}$ of complex numbers is a bilinear map which for every $\alpha \in \mathbb{C}^n$, and $\beta \in \mathbb{C}^n$, in addition to bilinearity property, satisfies three conditions

$$(\alpha, \beta) \;=\; (\beta, \alpha)^*$$

$$(\alpha, \alpha) \;\geq\; 0$$

and

$$(\alpha, \alpha) = 0 \quad \Longrightarrow \quad \alpha = 0.$$

Recall that $c = (\alpha, \beta)$ is a complex number, $c = a + ib$ and that $c^* = a - ib$ with $i = \sqrt{-1}$.

The inner product in a finite-dimensional vector space induces a norm, but a norm may exist even if a inner product is not defined. A finite dimensional vector space with a norm is a Banach space.

The inner product permits to measure the "length" of a vector and the "angle" between two vectors. If $(\alpha, \beta) \in \mathbb{C}^n$ and $c \in \mathbb{C}$, then the norm is a non-negative function with the following properties

$$|| \; \alpha + \beta \; || \;\; \leq \;\; || \; \alpha \; || + || \; \beta \; ||$$

$$|| \; c \cdot \alpha \; || \;\; = \;\; | \; c \; | \;\; || \; \alpha \; ||$$

and

$$|| \; \alpha \; || \;\; = \;\; 0 \quad \text{if and only if} \quad \alpha = 0.$$

We define orthogonality and orthonormal vector bases, in an n-dimensional Euclidean vector space over the field of complex numbers similarly to the ones defined for an n-dimensional Euclidean vector space over the field of real numbers.

We now introduce Hermitian operators, a concept analogous to the symmetric bilinear form in a real n-dimensional Euclidean space. If $\alpha, \beta \in \mathbb{C}^n$ then the bilinear form $\mathbf{A}(\alpha; \beta)$ is called *Hermitian* if

$$\mathbf{A}(\alpha; \beta) = \mathbf{A}^*(\beta; \alpha).$$

A bilinear form has a matrix associated with it and the necessary and sufficient condition for an operator $\mathbf{A}$ to be Hermitian is that its matrix $A = [a_{ij}]$ relative to some basis $e_1, e_2 \ldots e_n$ satisfies the condition

$$a_{ij} = a_{ji}^*.$$

The *adjoint operator* associated with a bilinear form $\mathbf{A}$ is denoted as $\mathbf{A}^\dagger(\alpha; \beta) = \mathbf{A}^*(\beta; \alpha)$. If $\alpha \in \mathbb{C}^n$ then by definition $\alpha^\dagger = (\alpha^*)^T$.

If A is the matrix representation of the linear operator $\mathbf{A}$ and we want to obtain the adjoint matrix, we first construct the complex conjugate matrix A^* and then take its transpose

$$A^\dagger = (A^*)^T.$$

For example, the adjoint of matrix

$$A = \begin{pmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{pmatrix}$$

is

$$A^{\dagger} = \begin{pmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{pmatrix}^{\dagger} = \left[\begin{pmatrix} 1-5i & 1+i \\ 1+3i & 7i \end{pmatrix}^{*} \right]^{T}.$$

Thus,

$$A^{\dagger} = \begin{pmatrix} 1+5i & 1-i \\ 1-3i & -7i \end{pmatrix}^{T} = \begin{pmatrix} 1+5i & 1-3i \\ 1-i & -7i \end{pmatrix}.$$

The adjoint of the matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$I_3^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the adjoint of matrix A with real elements, $(a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p) \in \mathbb{R}$

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

is

$$A^{\dagger} = \begin{pmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{pmatrix}.$$

An operator $\mathbf{A}$ is *normal* if

$$\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}.$$

If $\mathbf{A}$ is a Hermitian (self-adjoint) operator, then it is also a normal operator. A matrix U is *unitary* if

$$U^{\dagger}U = I_n$$

where I_n is the identity matrix in an n-dimensional vector space

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Unitary operators preserve the inner product of vectors. Let $(\alpha, \beta) \in \mathbb{C}^n$. Then

$$(U\alpha, U\beta) = (\alpha, \beta).$$

Usually we write the inner product using the "$\cdot$" symbol. With this convention the previous equation becomes

$$U\alpha \cdot U\beta = \alpha \cdot \beta.$$

Given two operators $\mathbf{A}$ and $\mathbf{B}$, the *commutator* of the two operators is

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} \; - \; \mathbf{BA}.$$

The operator $\mathbf{A}$ commutes with $\mathbf{B}$ if

$$[\mathbf{A}, \mathbf{B}] = 0.$$

The *anti-commutator* of $\mathbf{A}$ and $\mathbf{B}$ is

$$\{\mathbf{A}, \mathbf{B}\} = \mathbf{AB} \; + \; \mathbf{BA}.$$

The operator $\mathbf{A}$ anti-commutes with $\mathbf{B}$ if

$$\{\mathbf{A}, \mathbf{B}\} = 0.$$