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Laplace Transform and its application for solving differential equations. Fourier and Z Transforms

Motivation. Transform methods are widely used in many areas of science and engineering. For example, transform methods are used in signal processing and circuit analysis, in applications of probability theory. The basic idea is to transform a function from its original domain into a transform domain where certain operations can be carried out more efficiently, carrying out the operation in the transform domain, and then carrying out an inverse transform of the result (from the transform domain to the original domain).

For example, the *convolution* operation of two functions of time t , $f(t)$ and $g(t)$ is defined as:

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) d\tau = \int_{-\infty}^{+\infty} f(t - \tau) \cdot g(\tau) d\tau$$

with τ a real number. The convolution in the time domain becomes multiplication in the Laplace or Fourier domain. As an application, consider a linear circuit with the impulse response $h(t)$ and with the signal $x(t)$ as input. Then the output of the linear circuit is $y(t) = x(t) * h(t)$. If $H(s)$, $X(s)$ and $Y(s)$ are the Laplace transforms of the impulse response, the input, and the output, respectively, then $Y(s) = H(s) \cdot X(s)$, as seen in Figure 1. Once we know $Y(s)$ we can apply the inverse Laplace Transform to obtain the response of the circuit, $y(t)$, function of time t .

Transform methods provide a bridge between the commonly used method of separation of variables and numerical techniques for solving linear partial differential equations. While in some ways similar to separation of variables, transform methods can be effective for a wider class of problems. Even when the inverse of the transform cannot be found analytically, numeric and asymptotic techniques now exist for their inversion.

Laplace Transform. Let \mathbb{R} be the field of real numbers and \mathbb{C} the field of complex numbers. Consider a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(t), t \in \mathbb{R}, t \geq 0$. Then the *Laplace Transform* of $f(t)$ is denoted as $\mathcal{L}[f(t)]$ and it is defined as $F(s)$ with $s \in \mathbb{C}$:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.$$

The Laplace transform $F(s)$ typically exists for all complex numbers s such that $Re(s) > a$ where $a \in \mathbb{R}$ is a constant which depends on the behavior of $f(t)$.

The *Inverse Laplace Transform* is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s) ds$$

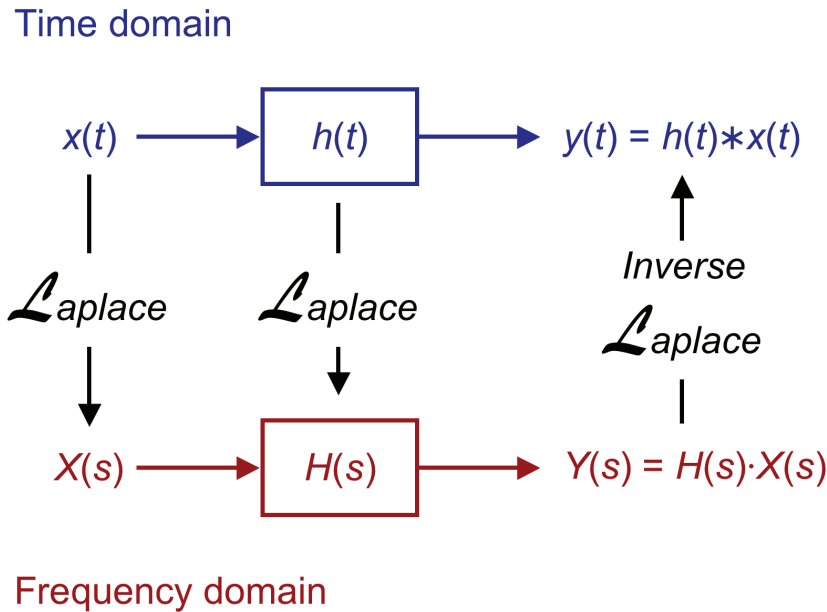


Figure 1: A circuit with the impulse response $h(t)$ and $x(t)$ as input. Then the output is $y(t) = x(t) * h(t)$. If $H(s)$, $X(s)$ and $Y(s)$ are respectively the Laplace transforms of the impulse response, the input, and the output, then $Y(s) = H(s) \cdot X(s)$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$ normally requiring $s_\rho > \text{Re}(s_\rho)$ for every singularity s_ρ of $F(s)$ and with $i = \sqrt{-1}$. If all singularities are in the left half-plane (in this case $\text{Re}(s_\rho) < 0$ for every s_ρ), then γ can be set to zero and the above inverse integral formula becomes identical to the inverse Fourier transform. This integral is known as the Fourier-Mellin integral.

The *Bilateral Laplace Transform* is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(t) dt.$$

In probability theory, the Laplace transform is defined by means of an expectation value. If X is a random variable with probability density function f_X , then the Laplace transform of f_X is given by the expectation:

$$(\mathcal{L}f_X)(s) = E[e^{-sX}].$$

By abuse of language, one often refers instead to this as the Laplace transform of the random variable X . Replacing the variable s by $-t$ gives the moment generating function of X . The Laplace transform has applications throughout probability theory, including first passage times of stochastic processes such as Markov chains and renewal theory.

Laplace Transforms of a few functions $f(t)$. In each case we start from the definition. For example, $f(t) = k$:

$$F(s) = \mathcal{L}[k] = \int_0^{\infty} e^{-st} k dt = -\frac{k}{s} [e^{-st}]_{t=0}^{t \rightarrow \infty} = \frac{k}{s}.$$

Now $f(t) = e^{\alpha t}$

$$F(s) = \mathcal{L}[e^{\alpha t}] = \int_0^{\infty} e^{-st} e^{\alpha t} dt = -\frac{1}{s-\alpha} [e^{-(s-\alpha)t}]_{t=0}^{t \rightarrow \infty} = \frac{1}{s-\alpha}.$$

Properties of the Laplace Transform. The properties of the Laplace Transform summarized in Table 1 can be derived easily starting from the definition.

Table 1: Properties of the Laplace Transform. The function $f(t)$ is assumed to be n -times differentiable, with n -th derivative of exponential type. Notations: $F(s) = \mathcal{L}[f(t)]$, $G(s) = \mathcal{L}[g(t)]$, $f^{(n)}$ is the n -th derivative of $f(t)$, $F^{(n)}(s)$ is the n -th derivative of $F(s)$, $u(t)$ is the Heaviside step function, $u(t) = \int_{-\infty}^t \delta(\tau) d\tau$ with δ the Dirac delta function, $(f * g)(t)$ is the convolution of $f(t)$ and $g(t)$, $\alpha, \beta \in \mathbb{R}$.

Property	Time-domain	s-domain
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Scaling	$f(\alpha t)$	$\frac{1}{\alpha} F(\frac{s}{\alpha})$
Frequency shifting	$e^{\alpha t} f(t)$	$F(s - \alpha)$
Time shifting	$f(t - \alpha)u(t - \alpha)$	$e^{-\alpha s} F(s)$
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
Frequency integration	$f(t)/t$	$\int_s^{\infty} F(r) dr$
Differentiation	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
Integration	$\int_0^t f(\tau) d\tau$	$(u * f)(t) \frac{1}{s} F(s)$
Convolution	$(f * g)(t)$	$F(s) * G(s)$
Periodic function $f(t) = f(t + T)$	$f(t)$	$\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t) dt$

For example, to prove linearity consider two functions $f(t)$ and $g(t)$ and their Laplace Transforms:

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt \quad \text{and} \quad G(s) = \mathcal{L}[g(t)] = \int_0^{\infty} e^{-st} g(t) dt.$$

From the definition of the Laplace Transform it follows that

$$\mathcal{L}[f(t) + g(t)] = \int_0^{\infty} e^{-st} [f(t) + g(t)] dt = \int_0^{\infty} e^{-st} f(t) dt + \int_0^{\infty} e^{-st} g(t) dt = F(s) + G(s).$$

It is also easy to see that $F(0)$ represents the area under the curve $f(t)$:

$$F(s = 0) = \int_0^{\infty} f(t) dt$$

The Laplace Transform can be expressed as:

$$\mathcal{L}[f(t)] = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{f'''(0)}{s^4} + \dots$$

Proof: This important property of the Laplace Transform is a consequence of the following equality:

$$\int e^{-\alpha x} f(x) dx = -\frac{e^{-\alpha x}}{\alpha} \left[f(x) + \frac{f'(x)}{\alpha} + \frac{f''(x)}{\alpha^2} + \frac{f'''(x)}{\alpha^3} + \dots \right]$$

This is easy to prove by applying the derivation operator of both sides; then the left hand side becomes $A = e^{-\alpha x} f(x)$. The right hand side is the sum of two terms B and C :

$$B = \alpha \frac{e^{-\alpha x}}{\alpha} \left[f(x) + \frac{f'(x)}{\alpha} + \frac{f''(x)}{\alpha^2} + \frac{f'''(x)}{\alpha^3} + \dots \right]$$

$$C = -\frac{e^{-\alpha x}}{\alpha} \left[f'(x) + \frac{f''(x)}{\alpha} + \frac{f'''(x)}{\alpha^2} + \frac{f^{iv}(x)}{\alpha^3} + \dots \right]$$

Then

$$B + C = e^{-\alpha x} f(x).$$

Thus $A = B + C$ and this equality allows us to express the Laplace Transform as:

$$\mathcal{L}[f(t)] = \left\{ -\frac{e^{-st}}{s} \left[f(t) + \frac{f'(t)}{s} + \frac{f''(t)}{s^2} + \frac{f'''(t)}{s^3} + \dots \right] \right\}_0^\infty.$$

But:

$$\lim_{t \rightarrow \infty} \left\{ -\frac{e^{-st}}{s} \left[f(t) + \frac{f'(t)}{s} + \frac{f''(t)}{s^2} + \frac{f'''(t)}{s^3} + \dots \right] \right\} = 0$$

After we subtract the value of the expression for $t = 0$ we obtain the result enounced:

$$\mathcal{L}[f(t)] = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{f'''(0)}{s^4} + \dots$$

Corollary of this property: The Laplace Transform of the first and second derivatives of the function $f(t)$ can be expressed as:

$$\mathcal{L}[f'(t)] = sF(s) - f(0) \quad \text{and} \quad \mathcal{L}[f(t)''(t)] = s^2F(s) - sf(0) - f'(0)$$

with $F(s)$ the Laplace Transform of $f(t)$.

Proof: To compute $\mathcal{L}[f'(t)]$ we substitute $f'(t) = h(t)$ and observe that:

$$f''(t) = h'(t), \quad f'''(t) = h''(t), \dots$$

Using the property of the Laplace Transform:

$$H(s) = \mathcal{L}[h(t)] = \left[\frac{h(0)}{s} + \frac{h'(0)}{s^2} + \frac{h''(0)}{s^3} + \dots \right] = \left[\frac{f'(0)}{s} + \frac{f''(0)}{s^2} + \frac{f'''(0)}{s^3} + \dots \right].$$

Now write the expression of the Laplace Transform of the function $f(t)$ as:

$$F(s) = \mathcal{L}[f(t)] = \frac{f(0)}{s} + \frac{1}{s} \left[\frac{f'(0)}{s} + \frac{f''(0)}{s^2} + \frac{f'''(0)}{s^3} + \dots \right].$$

It follows immediately that:

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}H(s) \quad \text{or} \quad H(s) = sF(s) - f(0).$$

The second property can be proved in a similar way using the substitution $h(t) = f''(t)$ and writing:

$$F(s) = \mathcal{L}[f(t)] = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{1}{s^2} \left[\frac{f''(0)}{s} + \frac{f'''(0)}{s^2} + \dots \right].$$

Thus:

$$F(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{1}{s^2}H(s) \quad \text{or} \quad H(s) = s^2F(s) - sf(0) - f'(0).$$

Applications of the Laplace Transform for solving differential equations. The Laplace Transform allows us to transform ordinary differential equations into algebraic equations. For example consider a continuous function $f(t)$ with derivatives $f'(t)$ and $f''(t)$. A second order differential equation with the initial conditions $f(0) = \gamma$ and $f'(0) = \delta$ is:

$$af''(t) + bf'(t) + cf(t) = 0$$

If we apply the Laplace Transform:

$$\mathcal{L}[af''(t) + bf'(t) + cf(t)] = 0$$

Let $F(s)$ be the Laplace transform of $f(t)$. Using the linearity property and the expressions of the Laplace transform of the first and second derivative of the function $f(t)$ this equation becomes:

$$a[s^2F(s) - sf(0) - f'(0)] + b[sF(s) - f(0)] + cF(s) = 0$$

We want to determine $F(s)$ first and then determine the function $f(t) = \mathcal{L}^{-1}[F(s)]$:

$$F(s)[as^2 + bs + c] = asf(0) + af'(0) + bf(0)$$

thus:

$$F(s) = \frac{asf(0) + af'(0) + bf(0)}{as^2 + bs + c}$$

Let s_1 and s_2 be the roots of the second degree equation in s :

$$as^2 + bs + c = 0 \quad \text{or} \quad (s - s_1)(s - s_2) = 0$$

with:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then we can express:

$$F(s) = \frac{af(0)s + af'(0) + bf(0)}{(s - s_1)(s - s_2)}$$

The function $f(t)$ with the Laplace transform $F(s)$ is:

$$f(t) = Ae^{s_1 t} + Be^{s_2 t}$$

with:

$$A + B = af(0) \quad \text{and} \quad As_2 + Bs_1 = -[af'(0) + bf(0)].$$

It is easy to show these conditions using the properties of the Laplace Transform and the calculations of the Laplace transforms of an exponential function:

$$\mathcal{L}[f(t)] = \mathcal{L}[Ae^{s_1 t} + Be^{s_2 t}] = A\mathcal{L}[e^{s_1 t}] + B\mathcal{L}[e^{s_2 t}] = A\frac{1}{s - s_1} + B\frac{1}{s - s_2}$$

or

$$F(s) = \frac{s(A + B) - (As_2 + Bs_1)}{(s - s_1)(s - s_2)} = \frac{s[af(0)] + [af'(0) + bf(0)]}{(s - s_1)(s - s_2)}.$$

If we equate the coefficients of the polynomial in s at the numerator we get the expressions we expect:

$$A + B = af(0) \quad \text{and} \quad As_2 + Bs_1 = -[af'(0) + bf(0)].$$

Application of the Laplace Transform to the bungee jumper problem. We now consider the following model for $v(t)$, the velocity of the jumper function of time:

$$v'(t) = g - \frac{c}{m}v(t)$$

with g the gravitational constant and m the mass of the jumper, c a linear drag coefficient, and the initial velocity $v(0) = 0$. If $V(s) = \mathcal{L}[v(t)]$ then we can transform this first order differential equation into an algebraic one:

$$sV(s) - v(0) = \frac{g}{s} - \frac{c}{m}V(s).$$

It follows that:

$$V(s)(s + \frac{c}{m}) = \frac{g}{s} + v(0)$$

or

$$V(s) = \frac{g}{s(s + \frac{c}{m})} + \frac{v(0)}{s + \frac{c}{m}}$$

A technique called *partial fraction expansion* allows us to express the first term on the right hand side of the equations as follows:

$$\frac{g}{s(s + \frac{c}{m})} = \frac{A}{s} + \frac{B}{(s + \frac{c}{m})}.$$

The coefficients A and B can be determined easily from the identity:

$$\frac{g}{s(s + \frac{c}{m})} \equiv \frac{s(A + B) + A\frac{c}{m}}{s(s + \frac{c}{m})}$$

as

$$A + B = 0 \quad \text{and} \quad A\frac{c}{m} = g.$$

Thus:

$$A = \frac{gm}{c} \quad \text{and} \quad B = -\frac{gm}{c}.$$

Thus the Laplace Transform of the velocity is:

$$V(s) = \frac{gm}{sc} - \frac{gm}{c(s + \frac{c}{m})} + \frac{v(0)}{s + \frac{c}{m}}$$

To find the velocity, $v(t)$ we apply the Inverse Laplace Transform to individual terms in the equation giving $F(s)$:

$$\mathcal{L}^{-1} \left[\frac{gm}{sc} \right] = \frac{gm}{c}.$$

$$\mathcal{L}^{-1} \left[-\frac{gm}{c(s + \frac{c}{m})} \right] = -\frac{gm}{c} \times \frac{1}{s + \frac{c}{m}} e^{-\frac{c}{m}t}.$$

$$\mathcal{L}^{-1} \left[\frac{v(0)}{s + \frac{c}{m}} \right] = v(0)e^{-\frac{c}{m}t}.$$

Finally:

$$v(t) = \mathcal{L}^{-1} [V(s)] = \frac{gm}{c} - \frac{gm}{c} \times \frac{1}{s + \frac{c}{m}} e^{-\frac{c}{m}t} + v(0)e^{-\frac{c}{m}t}.$$

Continuous Fourier Transform. When s is an imaginary number, $s = i\omega$, or $s = i2\pi f$; $\omega = 2\pi f$ is called the angular frequency, $f = 1/T$ the frequency, and T is the period of the function. Then the Laplace transform of the function $q(t)$ becomes the continuous Fourier transform, $\mathcal{F}[q(t)]$:

$$Q(f) = \mathcal{L}[q(t)]|_{s=i2\pi f} = \mathcal{F}[q(t)] = \int_{-\infty}^{\infty} e^{-i2\pi ft} q(t) dt.$$

Fourier analysis is widely used in signal processing. The spectrum $S(f)$ of the signal $s(t)$, is the Fourier Transform of $s(t)$ and has a meaningful physical interpretation.

z-Transform. Consider a function $f(t)$. Call $\Delta_T(t)$ a sampling function with period T and frequency $f = 1/T$. $\Delta_T(t)$ is defined as:

$$\Delta_T(t) = \sum_{n=0}^{\infty} \delta(t - nT).$$

Sampling the continuous function $f(t)$ means to observe the function at discrete instances of time, in other words to record a discrete set of samples of the function, $f[n] = f(nT)$. Call $f_s(t)$ the *sampled function*:

$$f_s(t) = f(t) \cdot \Delta_T(t) = f(t) \sum_{n=0}^{\infty} \delta(t - nT) = \sum_{n=0}^{\infty} f(nT) \delta(t - nT) = \sum_{n=0}^{\infty} f[n] \delta(t - nT).$$

The Laplace transform of the sampled function $f_s(t)$ is:

$$F_q(s) = \int_0^{\infty} f_s(t)e^{-st}.$$

Let us transform this expression:

$$\int_0^{\infty} f_s(t)e^{-st} = \int_0^{\infty} \sum_{n=0}^{\infty} f[n]\delta(t - nT)e^{-st} dt = \sum_{n=0}^{\infty} f[n] \int_{-0}^{\infty} \delta(t - nT)e^{-st} dt = \sum_{n=0}^{\infty} f[n]e^{-nsT}.$$

If we substitute $z = e^{sT}$ we have:

$$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n}.$$