

CDA6530: Performance Models of Computers and Networks

Review of Transform Theory

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Why using transform?
 Make analysis easier
 Two transforms for probability
 Non-negative integer r.v.
 Z-transform (or called probability generating function (pgf))
 Non-negative, real valued r.v.
 Laplace transform (LT)



Z-transform

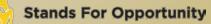
Definition: G_X(Z) is Z-transform for r.v. X

$$G_X(z) \equiv E[z^X] = \sum_{k=0}^{\infty} p_k z^k$$

Example: X is geometric r.v., p_k = (1-p)p^k

$$G_X(z) = \sum_{k=0}^{\infty} (1-p)p^k z^k = \frac{1-p}{1-pz},$$

For pz<1





• Poisson distr., $p_k = \lambda^k e^{-\lambda}/k!$

$$G_X(z) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} z^k$$

$$= e^{-\lambda}\sum_{k=0}^{\infty} (\lambda z)^k/k!$$

$$= e^{-\lambda(1-z)}$$

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Benefit

$$\frac{dG_X(z)}{dz} = \sum_{k=1}^{\infty} kp_k z^{k-1}$$

$$\square \text{ Thus } E[X] = \sum_{k=1}^{\infty} kp_k z^{k-1} \Big|_{z=1} = \frac{dG_X(z)}{dz} \Big|_{z=1}$$

$$\frac{d^2 G_X(z)}{dz^2} \bigg|_{z=1} = E[X^2] - E[X]$$

Convolution: X, Y independent with Ztransforms G_X(z) and G_Y(z), Let U=X+Y

$$G_U(z) = G_X(z)G_Y(z)$$



Solution of M/M/1 Using Transform

$$\begin{aligned} &(\lambda+\mu)\pi_i &= \lambda\pi_{i-1}+\mu\pi_{i+1}, \quad i=1,\dots\\ &\lambda\pi_0 &= \mu\pi_1 \end{aligned}$$

• Multiplying by z^i , using $\rho = \lambda/\mu$, and summing over i

$$(1+\rho)\sum_{i=0}^{\infty}\pi_{i}z^{i} = \rho z\sum_{i=0}^{\infty}\pi_{i}z^{i} + z^{-1}\sum_{i=1}^{\infty}\pi_{i}z^{i} + \pi_{0}$$

$$(1+\rho)G_N(z) = \rho z G_N(z) + z^{-1}(G_N(z) - \pi_0) + \pi_0$$

$$(\rho z^{2} - (1+\rho)z + 1)G_{N}(z) = (1-z)\pi_{0}$$

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$$\rho z^2 - (1+\rho)z + 1 = (1-z)(1-\rho z) \implies$$

$$G_N(z) = \frac{\pi_0}{1 - \rho z},$$
$$= \frac{1 - \rho}{1 - \rho z}$$

Stands For Opportunity

UCF

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$$E[N] = \frac{dG_N(z)}{dz}|_{z=1}$$
$$= \frac{1-\rho}{(1-\rho z)^2}\rho|_{z=1}$$
$$= \frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$$

UCF Stands For Opportunity

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Laplace Transform

R.v. X has pdf f_X(x)
 X is non-negative, real value
 The LT of X is:

$$F_X^*(s) \equiv E[e^{-sX}] = \int_0^\infty f_X(x)e^{-sx}dx$$



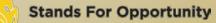
Example

• X: exp. Distr. $f_X(x) = \lambda e^{-\lambda x}$

$$F_X^*(s) = \int_0^\infty \lambda e^{-x(\lambda+s)} dx = \frac{\lambda}{\lambda+s}$$

Moments:

$$E[X] = -\frac{d}{ds} F_X^*(s) \Big|_{s=0}$$
$$E[X^i] = (-1)^i \frac{d^{(i)}}{ds} F_X^*(s) \Big|_{s=0}$$



Convolution

□ X₁, X₂, …, X_n are independent rvs with $F_{X_1}^*(s), F_{X_2}^*(s), \dots, F_{X_n}^*(s)$

□ If Y=X₁+X₂+···+X_n

$$F_Y^*(s) = F_{X_1}^*(s) \cdot F_{X_2}^*(s) \cdots F_{X_n}^*(s)$$

• If Y is n-th Erlang, $F_Y^*(s) = (\frac{\lambda}{\lambda + s})^n$

Z-transform and LT

□ X_1, X_2, \dots, X_N are i.i.d. r.v with LT $F_X^*(s)$ □ N is r.v. with pgf $G_N(z)$ □ Y= X₁+X₂+...+X_N

$$F_Y^*(s) = G_N(F_X^*(s))$$

• If X_i is discrete r.v. with $G_X(Z)$, then $G_Y(z) = G_N(G_X(z))$