UCF Stands For Opportunity

CDA6530: Performance Models of Computers and Networks

Review of Transform Theory

SCHOOL OF ELECTRICAL ENGINEERING & COMPUTER SCIENCE

- Why using transform?
  - Make analysis easier
- Two transforms for probability
  - Non-negative integer r.v.
    - Z-transform (or called probability generating function (pgf))
  - Non-negative, real valued r.v.
    - Laplace transform (LT)

Z-transform

- Definition:
  - $G_X(Z)$ is Z-transform for r.v. X

$$G_X(z) \equiv E[z^X] = \sum_{k=0}^{\infty} p_k z^k$$

- Example:
  - X is geometric r.v., $p_k = (1-p)p^k$

$$G_X(z) = \sum_{k=0}^{\infty} (1-p)p^k z^k = \frac{1-p}{1-pz},$$

For $pz<1$

- Poisson distr., $p_k = \lambda^k e^{-\lambda}/k!$

$$G_X(z) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} z^k$$

$$= \quad e^{-\lambda} \sum_{k=0}^{\infty} (\lambda z)^k / k!$$

$$= \quad e^{-\lambda(1-z)}$$

Benefit

$$\frac{dG_X(z)}{dz} = \sum_{k=1}^{\infty} k p_k z^{k-1}$$

- Thus $E[X] = \sum_{k=1}^{\infty} k p_k z^{k-1} \Big|_{z=1}$ $= \frac{dG_X(z)}{dz}\Big|_{z=1}$

$$\frac{d^2 G_X(z)}{dz^2}\Big|_{z=1} = E[X^2] - E[X]$$

- Convolution: X, Y independent with Z-transforms $G_X(z)$ and $G_Y(z)$, Let U=X+Y

$$G_U(z) = G_X(z) G_Y(z)$$

Solution of M/M/1 Using Transform

$$
\begin{aligned}
(\lambda+\mu)\pi_i &= \lambda\pi_{i-1}+\mu\pi_{i+1}, \quad i=1,\ldots \\
\lambda\pi_0 &= \mu\pi_1
\end{aligned}
$$

- Multiplying by z^i, using $\rho = \lambda/\mu$, and summing over i

$$(1+\rho)\sum_{i=0}^{\infty}\pi_i z^i = \rho z\sum_{i=0}^{\infty}\pi_i z^i + z^{-1}\sum_{i=1}^{\infty}\pi_i z^i + \pi_0$$

$$(1+\rho)G_N(z) = \rho z G_N(z) + z^{-1}(G_N(z)-\pi_0)+\pi_0$$

$$(\rho z^2-(1+\rho)z+1)G_N(z) = (1-z)\pi_0$$

$$\rho z^2-(1+\rho)z+1 = (1-z)(1-\rho z) \quad \Rightarrow \quad
\begin{aligned}
G_N(z) &= \frac{\pi_0}{1-\rho z}, \\
&= \frac{1-\rho}{1-\rho z}
\end{aligned}
$$

$$
\begin{aligned}
E[N] &= \frac{dG_N(z)}{dz}\Big|_{z=1} \\
&= \frac{1-\rho}{(1-\rho z)^2}\rho\Big|_{z=1} \\
&= \frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}
\end{aligned}
$$

Laplace Transform

- R.v. X has pdf $f_X(x)$
  - X is non-negative, real value
- The LT of X is:

$$F_X^*(s) \equiv E[e^{-sX}] = \int_0^\infty f_X(x) e^{-sx} dx$$

Example

- X: exp. Distr. $f_X(x)=\lambda e^{-\lambda x}$

$$F_X^*(s) = \int_0^\infty \lambda e^{-x(\lambda+s)} dx = \frac{\lambda}{\lambda+s}$$

- Moments:

$$E[X] = -\frac{d}{ds} F_X^*(s)\Big|_{s=0}$$

$$E[X^i] = (-1)^i \frac{d^{(i)}}{ds} F_X^*(s)|_{s=0}$$

Convolution

- $X_1, X_2, \cdots, X_n$ are independent rvs with

$$F^*_{X_1}(s), F^*_{X_2}(s), \cdots, F^*_{X_n}(s)$$

- If $Y = X_1 + X_2 + \cdots + X_n$

$$F^*_Y(s) = F^*_{X_1}(s) \cdot F^*_{X_2}(s) \cdots F^*_{X_n}(s)$$

- If Y is n-th Erlang,

$$F^*_Y(s) = (\frac{\lambda}{\lambda + s})^n$$

Z-transform and LT

- $X_1, X_2, \cdots, X_N$ are i.i.d. r.v with LT $F_X^*(s)$
- N is r.v. with pgf $G_N(z)$
- $Y = X_1 + X_2 + \cdots + X_N$

$$F_Y^*(s) = G_N(F_X^*(s))$$

- If X_i is discrete r.v. with $G_X(Z)$, then

$$G_Y(z) = G_N(G_X(z))$$