Motivation

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Motivation

• A limitation of active contours based on parametric curves of the form $f(s)$ (snakes, b-snakes,...) is that it is challenging to change the topology of the curve as it evolves.
• If the shape changes dramatically, curve reparameterization may also be required.
• An alternative representation for such closed contours is to use **level sets (LS)**.
  – LS evolve to fit and track objects of interest by modifying the underlying embedding function instead of curve function $f(s)$
Basics of Level Sets

Energy measures the consistency of the image statistics inside and outside the regions.

Cremers, Rousson, and Deriche 2007
Image Segmentation with Level Sets

- Contour evolution (*Sethian and Osher, 1988*)
- Level sets for closed contours
  - Zero-crossing(s) of a characteristic function define the curve
  - Fit and track objects of interest by modifying the underlying embedding function $\phi(x, y)$ instead of the curve $f(s)$
  - Efficient algorithm
    - A small strip around the locations of the current zero-crossing needs to be updated at each step

Fast Marching Methods
Moving Interfaces

- 2D Moving Curves
- 3D Moving Surfaces

Ex:
- Interfaces between water and oil
- Propagating front of bush fire
- Deformable elastic solid
Evolving Curves and Surfaces

- Propagate curve according to speed function \( v = F n \)
- \( F \) depends on space, time, and the curve itself
- Surfaces in three dimensions

Tangential motion does not change the interface!

Only velocity component normal to surface is important!
Describe curve as Level Sets of a Function

\[ \phi(x, y) = x^2 + y^2 - 1 = 0 \]

Isocontour is the unit circle (implicit representation.)
Describe curve as Level Sets of a Function

\[ \phi(x, y) = x^2 + y^2 - 1 = 0 \]

A few isocontours of two dimensional function (circle)
Along with some representative normals.

GRADIENT:

\[ \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) \]
Describe curve as Level Sets of a Function

Then, unit normal (outward) is

\[ \vec{N} = \frac{\nabla \phi}{|\nabla \phi|} \]
Describe curve as Level Sets of a Function

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On Cartesian grid, we need to approximate this equation (ex. Finite difference techniques):

\[ \frac{\partial \phi}{\partial x} \approx \frac{\phi_{i+1} - \phi_i}{\Delta x} \]
Describe curve as Level Sets of a Function

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Mean curvature of the interface is defined as the divergence of the normal \( \vec{N} = (n_1, n_2) \)

\[
\kappa = \nabla . \vec{N} = \frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y} = \nabla . \left( \frac{\nabla \phi}{|\nabla \phi|} \right)
\]
Describe curve as Level Sets of a Function

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Describe curve as Level Sets of a Function
Variational Formulations and LS

• Transition from Active Contours:
  – contour v(t) $\rightarrow$ front $\gamma(t)$
  – contour energy $\rightarrow$ forces $F_A, F_C$
  – image energy $\rightarrow$ speed function $k_i$

• Level set:
  – The level set $c_0$ at time $t$ of a function $\psi(x,y,t)$ is the set of arguments $\{ (x,y), \psi(x,y,t) = c_0 \}$
  – Idea: define a function $\psi(x,y,t)$ so that at any time,
    $$\gamma(t) = \{ (x,y), \psi(x,y,t) = 0 \}$$
  • there are many such $\psi$
  • $\psi$ has many other level sets, more or less parallel to $\gamma$
  • only $\gamma$ has a meaning for segmentation, not any other level set of $\psi$
Level Set Framework

Usual choice for $\psi$: signed distance to the front $\gamma(0)$

$$
\psi(x,y,0) = \begin{cases} 
- d(x,y, \gamma) & \text{if } (x,y) \text{ inside the front} \\
0 & \text{on} \\
d(x,y, \gamma) & \text{outside}
\end{cases}
$$

\[
\begin{array}{cccccccccccc}
7 & 6 & 5 & 4 & 4 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 \\
6 & 5 & 4 & 3 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\
5 & 4 & 3 & 2 & 2 & 2 & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 2 & 3 \\
4 & 3 & 2 & 1 & 1 & 1 & 0 & -1 & -2 & -2 & -2 & -1 & 0 & 1 & 2 \\
3 & 2 & 1 & 0 & 0 & 0 & -1 & -2 & -3 & -3 & -2 & -2 & -1 & 0 & 1 & 2 \\
2 & 1 & 0 & -1 & -1 & -1 & -2 & -3 & -3 & -2 & -2 & -1 & 0 & 1 & 2 & 3 \\
2 & 1 & 0 & -1 & -2 & -2 & -2 & -3 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
2 & 1 & 0 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
5 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 5 & 4 & 3 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]
Lecture 16: Deformable Models and Segmentation

Level Set Framework

- no movement, only change of values
- the front may change its topology
- the front location may be between samples

\[ \psi(x, y, t) + \Delta \psi(x, y, t) = \psi(x, y, t+1) \]

\[ \psi(x, y, t) \]
Level Set

Segmentation with LS:

• Initialise the front $\gamma(0)$
• Compute $\psi(x,y,0)$
• Iterate:
  $$\psi(x,y,t+1) = \psi(x,y,t) + \Delta \psi(x,y,t)$$
  until convergence
• Mark the front $\gamma(t_{\text{end}})$
Recap: Variational Formulations and LS

- Transition from Active Contours:
  - contour \( v(t) \) → front \( \gamma(t) \)
  - contour energy → forces \( F_A, F_C \)
  - image energy → speed function \( k_i \)

- Level set:
  - The level set \( c_0 \) at time \( t \) of a function \( \psi(x,y,t) \) is the set of arguments \( \{ (x,y) , \psi(x,y,t) = c_0 \} \)
  - Idea: define a function \( \psi(x,y,t) \) so that at any time,
    \[
    \gamma(t) = \{ (x,y) , \psi(x,y,t) = 0 \}
    \]
      - there are many such \( \psi \)
      - \( \psi \) has many other level sets, more or less parallel to \( \gamma \)
      - only \( \gamma \) has a meaning for segmentation, not any other level set of \( \psi \)
Front Propagation

\[ \frac{\partial \psi}{\partial t} + \left( k_I \right) \cdot \left( F_A + F_G(\kappa) \right) \cdot \left\| \nabla \psi \right\| = 0 \]

- **ψ(x,y,t+1) - ψ(x,y,t)**
  - extension of the speed function \( k_I \)
    - (image influence)

- **constant “force” (balloon pressure)**
  - \( \kappa = \text{div} \left( \frac{\nabla \psi}{\| \nabla \psi \|} \right) \)

- **kappa**
  - \( \kappa = \text{div} \left( \frac{\nabla \psi}{\| \nabla \psi \|} \right) \)
  - (contour influence)

- **smoothing “force” depending on the local curvature \( \kappa \)**
  - spatial derivative of \( \psi \)
  - product of influences

- **link between spatial and temporal derivatives, but not the same type of motion as contours!**

- \( \psi \) link between spatial and temporal derivatives, but not the same type of motion as contours!
Front Propagation

- Speed function:
  - $k_I$ is meant to stop the front on the object’s boundaries
  - similar to image energy: $k_I(x,y) = 1 / (1 + || \nabla I(x,y)||)$
  - only makes sense for the front (level set 0)
  - yet, same equation for all level sets
    → extend $k_I$ to all level sets, defining $\hat{k}_I$

- possible extension:

  $$\hat{k}_I(x,y) = k_I(x',y')$$

  where $(x',y')$ is the point in the front closest to $(x,y)$

  (such a $\hat{k}_I(x,y)$ depends on the front location)
LS Algorithm

1. compute the speed $k_I$ on the front
   extend it to all other level sets

2. compute $\psi(x,y,t+1) = \psi(x,y,t) + \Delta \psi(x,y,t)$

3. find the front location (for next iteration)
   modify $\psi(x,y,t+1)$ by linear interpolation

$\psi(x,y,t)$
Narrow Band Extension

• Weaknesses of algorithm 1
  – update of all $\psi(x,y,t)$: inefficient, only care about the front
  – speed extension: computationally expensive

• Improvement:
  – narrow band: only update a few level sets around $\gamma$
  – other extended speed: $k_i(x,y) = 1 / (1 + || \nabla I(x,y) ||)$
Narrow Band Extension

• Caution:
  – extrapolate the curvature $\kappa$ at the edges
  – re-select the narrow band regularly:
    an empty pixel cannot get a value
    $\rightarrow$ may restrict the evolution of the front
Summary of LS

• Level sets:
  – function $\psi : [0, I_{\text{width}}] \times [0, I_{\text{height}}] \times N \rightarrow \mathbb{R}$
  – $(x, y, t) \rightarrow \psi(x,y,t)$
  – embed a curve $\gamma$: $\gamma(t) = \{ (x,y), \psi(x,y,t) = 0 \}$
  – $\gamma(0)$ is provided externally, $\psi(x,y,0)$ is computed
  – $\psi(x,y,t+1)$ is computed by changing the values of $\psi(x,y,t)$
  – changes using a product of influences
  – on convergence, $\gamma(t_{\text{end}})$ is the border of the object

• Issue:
  – computation time (improved with narrow band)
Examples
Summary of Boundary Based Methods

• Curves: object boundaries

• **Intelligent scissors (live-wire)**
  – sketch in real time a curve that clings to object boundaries

• **Snakes**
  – energy-minimizing, 2D spline curve that evolves towards image features such as strong edges

• **Level set**
  – Evolve the curve as the zero-set of a characteristic function,
  – Easily change topology and incorporate region-based statistics
Applications: Contour Tracking and Rotoscoping

- Track facial features for performance-driven animation
- Track heads and people, moving vehicles
- Medical image segmentation (CT image)
- Rotoscoping

Agarwala, Hertzmann, Seitz et al. (2004)
Applications: Image Restoration

SNR is approximately 3

Original Image
Applications: Image Restoration
Applications: Reconstruction of Surfaces from Unorganized Data Points

Reconstruction of a rat brain from data of MRI slices
• Localized appearance, top-down shape information and level set were combined to segment and track people in videos.
CVPR 2005: Level Set Segmentation (1800 citation)

- Ultrasound image segmentation.

- Chunming Li et al. LS Evolution without reinitialization: a new variational formulation.
Application: Vein Segmentation
Slice Credits and References

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- Osher and Fedkiw. Level set methods and dynamic implicit surfaces.
- Lim, Bagci, and Li. IEEE TBME 2013 [Willmore Flow and Level Set]