Computer Science Foundation Exam

August 12, 2016

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

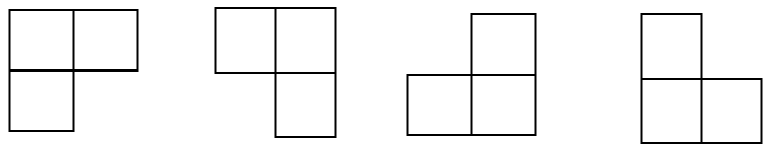
Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	15	PRF (Logic)	10	
3	10	PRF (Sets)	7	
4	10	NTH (Number Theory)	7	
ALL	50		34	

You must do all 4 problems in this section of the exam.

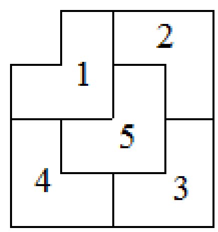
Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) PRF (Induction)

A tromino is a tile consisting of three unit squares in an L shape. The following are the four possible orientations a tromino can be placed:



Using induction on n, prove that for all non-negative integers, n, a 2^n x 2^n grid of unit squares with a single unit square removed can be tiled properly with a set of trominos. A proper tiling covers every unit square of the original object with a single unit square of a single tromino. For example, the following is a valid tiling of the 4 x 4 grid with the top left corner missing:



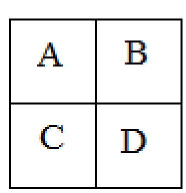
Solution

Base case: n=0, in this case we have a 2^0 x 2^0 square with one square missing, which means that what remains to be tiled is nothing, since we have one square missing from one square. Trivially, we can tile nothing with 0 L tiles.

Inductive hypothesis: For an arbitrary non-negative integer n = k, we can tile a 2^k x 2^k grid of unit squares with one square missing.

Inductive step: Prove for n = k+1 that we can tile a 2^{k+1} x 2^{k+1} grid of unit squares with one square missing.

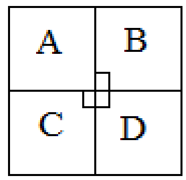
Given our square to tile, we partition it into 4 equal quadrants:



Note that since each quadrant is equal in size, each quadrant MUST BE a 2^k x 2^k square, since $2^k + 2^k = 2^{k+1}$. The missing unit square (that shouldn't be tiled) must be located in one of the four quadrants.

According to our inductive hypothesis, we can tile this quadrant (since it has a missing unit square).

Now, we have three quadrants left to tile. It must be the case that these three quadrants are "next to each other." For example, consider the case that the three quadrants are B, C and D. If this is the case, then we can place a single tromino at the center of the diagram above, with the tromino covering one unit square in B, one unit square in C and one unit square in D:



Now, we have tiled all of A with its square missing, as well as one unit square in quadrants B, C and D. Finally, we are left to tile quadrants B, C and D with ***exactly one unit square missing!!!*** We can perform this tiling based on the inductive hypothesis, completing the tiling of our original 2^{k+1} x 2^{k+1} design with a single unit square missing, completing the proof.

Grading: Base case - 2 pts, IH - 2 pts, IS - 2 pts, quadrant idea 2 pts, noting that one of these can naturally be tiled using the IH - 2 pts, placing a tile in the middle of the other three quadrants - 3 pts, use of IH to tile rest - 2 pts

2) (15 pts) PRF (Logic)

(a) (8 pts) Complete the truth table below.

p	q	r	$p \wedge q$	$\bar{p} \vee r$	$\overline{\bar{p} \vee r}$	$(p \wedge q) \vee (\overline{\bar{p} \vee r})$
F	F	F	**F**	**T**	**F**	**F**
F	F	T	**F**	**T**	**F**	**F**
F	T	F	**F**	**T**	**F**	**F**
F	T	T	**F**	**T**	**F**	**F**
T	F	F	**F**	**F**	**T**	**T**
T	F	T	**F**	**T**	**F**	**F**
T	T	F	**T**	**F**	**T**	**T**
T	F	T	**T**	**T**	**F**	**T**

Grading: 1 pt for each row, all or nothing. All 4 values on the row have to be completely correct to get credit for that row. Accept 0 = false, 1 = true as well.

(b) (7 pts) Create a logical expression using the variables p and q and only the logical operators $(\wedge)$ and $(\overline{\ \ })$ to create an expression which evaluates as described by the truth table below. (Note: There are many correct answers and each variable and operator may appear in the expression you create as many times as necessary.)

p	q	result
F	F	F
F	T	T
T	F	T
T	T	F

$$\overline{(\boldsymbol{p} \wedge \boldsymbol{q})} \wedge \overline{(\bar{\boldsymbol{p}} \wedge \bar{\boldsymbol{q}})}$$

Note: There are many possible answers, please check each response by hand.

Grading: 3 pts for following the rules and only using and and not.
1 pt for each row that matches the given truth table for their expression

3) (10 pts) PRF (Sets)

Let A, B and C be finite sets such that $A \subseteq B$, $B \subseteq A \cup C$, and $C \subseteq B$. Prove or disprove the following assertion: $|A| = |B|$ or $|A| = |C|$ or $|B| = |C|$.

This claim is false. Here is a counter-example which satisfies the given requirements with three sets of different sizes:

A = {1}
B = {1, 2, 3}
C = {2, 3}

In this example, we see that both A and C are subsets of B and that B is a subset of $A \cup C$, satisfying the given requirements. Yet, the cardinality of all three sets is different.

Intuitively, we are simply showing that if the union of two sets (A and C) equals a third set (B), which is what the given information forces, then while B is at least as large as A and C, there are no definitive requirements on the sizes of A and C.

Grading: Any proof gets 1 pt maximum. 2 pts for stating the claim is false. 5 pts for a valid counter-example, 3 pts for explaining why the counter-example is valid. If the counter-example is not valid, you may still give some partial credit out of the 5 pts allotted for the counter-example.

4) (10 pts) NTH (Number Theory)

Find an integer, n, in between 0 and 231, inclusive, such that $105n \equiv 1 \pmod{232}$. (Note: To earn full credit you must use the Extended Euclidean Algorithm.)

Start with the Euclidean algorithm to find the gcd of 232 and 105:

232 = 2 x 105 + 22
105 = 4 x 22 + 17
22 = 1 x 17 + 5
17 = 3 x 5 + 2
5 = 2 x 2 + 1
2 = 2 x 1, thus the gcd is 1.

Now, we run the Extended Euclidean Algorithm:

5 - 2 x 2 = 1
5 - 2(17 - 3 x 5) = 1
5 - 2 x 17 + 6 x 5 = 1
7 x 5 - 2 x 17 = 1
7(22 - 17) - 2 x 17 = 1
7 x 22 - 7 x 17 - 2 x 17 = 1
7 x 22 - 9 x 17 = 1
7 x 22 - 9 x(105 - 4x22) = 1
7 x 22 - 9 x 105 + 36 x 22 = 1
43 x 22 - 9 x 105 = 1
43(232 - 2 x 105) - 9 x 105 = 1
43 x 232 - 86 x 105 - 9 x 105 = 1
43 x 232 - 95 x 105 = 1

Taking this equation mod 232 we see that $-95 \times 105 \equiv 1 \pmod{232}$. Thus one solution for n that satisfies the given equation is -95. But, this value isn't in the range given. Any value equivalent to -95 mod 232 satisfies the equation, thus the correct answer given the restriction on n is -95 + 232 = 137.

Grading: 3 pts for regular Euclidean Algorithm, 6 pts for extended and getting -95, 1 pt for noting that 137 is equivalent to -95 and is the correct answer for the range given.

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Section II B

DISCRETE STRUCTURES

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SOLUTION

Question	Max Pts	Category	Passing	Score
1	10	CTG (Counting)	7	
2	10	PRB (Probability)	7	
3	15	PRF (Functions)	10	
4	15	PRF (Relations)	10	
ALL	50		34	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (10 pts) CTG (Counting)

a) (6 pts) In an election with 142,070,000 eligible voters and only three candidates to choose from for some particular office, how many different distributions are possible for the number of votes each candidate could receive, provided that every eligible voter is forced to vote, and they must vote for one of the three candidates (so, the voters can't abstain from voting or choose some write-in candidate)?

For example, if candidate A receives 100,000,000 votes, candidate B receives 32,070,000 votes, and candidate C receives 10,000,000 votes, that is different from A receiving 32,070,000 votes, B receiving 100,000,000 votes, and C receiving 10,000,000 votes.

Note that votes are cast anonymously, so all that matters is the number of votes each candidate receives, with no consideration for which voters those votes came from.

This is a classical stars and bars problem. We have 142,070,000 stars, and we want to throw down 2 bars to create 3 partitions. It's possible to have empty partitions, so from our 142,070,002 objects (stars + bars), we want to choose 2 to be bars (or, equivalently, 142,070,000 to be stars).

The answer is: C(142,070,002, 2)

Grading: +3 for recognizing this as combinations / stars and bars. +2 for identifying n = 142,070,002 and +1 for k = 2. (Award partial credit for n and/or k if the values are very close.)

b) (4 pts) What would be the answer to (a) if instead of voters being forced to vote, they were allowed to sit at home and not vote for any of the candidates? (But still, no write-in candidates are allowed on the ballot. Those who vote are constrained to the three candidates on the ballot.)

In this case, we need to create a fourth partition for all the discarded votes. So, we need three bars this time. The answer becomes: C(142,070,003, 3)

Grading: +2 for incrementing n by one, and +2 for incrementing k by one.

2) (10 pts) PRB (Probability)

Suppose six wizards are seated in a row along one side of a long, straight banquet table, in totally random order. Among those wizards are Lily Evans, James Potter, and Severus Snape.

Let P be the event that Lily Evans and James Potter end up sitting next to one another, and S the event that Severus Snape and Lily Evans end up sitting next to one another. Prove or disprove that P and S are independent events. (You may assume there are no magical shenanigans at play that would affect the probabilities of these events.)

First, to make this solution clear, let's number our seats like so:

______	______	______	______	______	______
S_1	S_2	S_3	S_4	S_5	S_6

Note that there are 6! ways to seat our six wizards: | S | = 6!.

For event P, we must choose a pair of seats for Lily and James to occupy. There are 5 ways to choose adjacent seats: S_1 and S_2, S_2 and S_3, and so on, up to S_5 and S_6. We can then seat Lily and James in those seats in two ways: either Lily is to the left of James, or James it to the left of Lily. We can seat the four remaining wizards in the four remaining seats in 4! Ways.

So, | P | = 5 · 2 · 4!, which means p(P) = 5 · 2 · 4! / 6! $= \frac{1}{3}$.

Similarly, | S | = $\frac{1}{3}$.

Now we must find p(P∩S). For both James and Severus to sit next to Lily, we must first choose three adjacent seats for them to occupy. There are 4 ways to do that. Then, we must seat Lily in the middle seat, and either James is to her left (and Severus to her right), or James is to her right (and Severus to her left). So, there are 2 ways to arrange the three wizards within their three adjacent seats. Finally, we can seat the three remaining wizards in 3! possible ways.

So, | P∩S | = 4 · 2 · 3!, which means p(P∩S) = 4 · 2 · 3! / 6! = $\frac{1}{15}$.

Since p(P∩S) = $\frac{1}{15}$ and p(P)·p(S) = $\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \neq \frac{1}{15}$, the events are ***not*** independent. (That is to say, they are dependent.)

Grading: +2 for the cardinality of the sample space, +3 for the probability of P (which is also the probability of S), +3 for the probability of P∩S, and +2 for showing the events are not independent by establishing the relationship between p(P∩S) and p(P)·p(S).

Note: If they compared p(P∩S) and p(P)·p(S), but thought that the inequality meant the results were independent, only award +2 for that part of the grading.

Please award partial credit as appropriate for alternate (but equally correct) solutions.

3) (15 pts) PRF (Functions)

Let $f: \mathbb{Z} \to \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$, where $f(x) = 5x + 10$ and $g(x) = 10x + 5$. Then:

(a) (3 pts) Give $f \circ g$.

$f \circ g = f(g(x)) = 5(10x + 5) + 10 = 50x + 25 + 10 = 50x + 35$

Grading: +3 for correct answer. +2 if they gave $g \circ f$ instead of $f \circ g$. +2 if they had a simple algebraic mistake.

(b) (4 pts) Prove or disprove that $f \circ g$ is surjective.

Disproof by counterexample: There is no $x \in \mathbb{Z}$ such that $(f \circ g)(x) = 1$. If there were, we would have $50x + 35 = 1$, which implies $x = -34/50$, which is not an integer.

Grading: +4 for a valid counterexample. +1 if they try to prove this is true and use good form to do so (even though the result is incorrect). Award partial credit in other cases you encounter as necessary.

(c) (4 pts) Prove or disprove that $f \circ g$ is injective.

Proof: Let $(f \circ g)(x_1) = (f \circ g)(x_2)$. Then we have:

$$50x_1 + 35 = 50x_2 + 35$$

$$50x_1 = 50x_2$$

$$x_1 = x_2$$

Thus, $f \circ g$ is injective.

Grading: +4 for showing that $(f \circ g)(x_1) = (f \circ g)(x_2) \Rightarrow x_1 = x_2$. Award partial credit if they are on the right track but have minor errors.

(d) (4 pts) For a function $h: \mathbb{Z} \to \mathbb{Z}$, use quantifiers to write a statement in symbolic logic that says h is a surjective function.

There are many valid solutions here. One of them is: $\forall y \in \mathbb{Z}\ (\exists x \in \mathbb{Z}\ (h(x) = y))$

Grading: +4 for a correct solution, +3 for something very close, +2 if they have the right idea for surjectivity but their use of quantifiers has errors.

4) (15 pts) PRF (Relations)

(a) (3 pts) What three properties must a relation satisfy in order to be an equivalence relation?

It must be reflexive, transitive, and symmetric.

Grading:
-1 pt for each incorrect property listed
-1 pt for each correct property missing from the list

(b) (6 pts) Is it possible to define an equivalence relation R on $A = \{1, 2, 3, 4, 5, 6, 7\}$ such that $|R|$ is even? If so, give one such equivalence relation. If not, *briefly* explain why not.

No, it's not possible for $|R|$ to be even if it is an equivalence relation on A. We know that R must contain the seven ordered pairs from the set $\{(x, x) \mid x \in A\}$, giving it odd cardinality initially. All other pairs that we could add to R are of the form (x, y), where $x \neq y$. If we add such a pair to R, then we must also add the pair (y, x) in order to keep R symmetric. Thus, we always add an even number of pairs to R, meaning that its cardinality always remains odd.

Grading: 2 pts for saying "no." 4 pts for a reasonable explanation. Award partial credit as necessary.

(c) (6 pts) Suppose we define a relation R by choosing 9 random ordered pairs (without replacement) from $A \times A$, where $A = \{1, 2, 3, 4, 5, 6, 7\}$. What is the probability that R will be an equivalence relation? (You must show your work.)

For R to be an equivalence relation, we must select for inclusion all the ordered pairs from the set $\{(x, x) \mid x \in A\}$. We then get to select two additional pairs, and they must be symmetric images of one another. (For example, if we select $(1, 2)$, we must also select $(2, 1)$ in order for R to remain symmetric, which would thereby ensure R remained an equivalence relation.)

There are $7 \cdot 7 = 49$ ordered pairs to choose from in $A \times A$. We note that order does not matter when choosing these pairs for our relation. So, there are C(49, 9) ways to select ordered pairs to be in our relation.

The number of ways to select 9 pairs while keeping R an equivalence relation is simply the number of ways to choose ordered pairs (x, y) and (y, x), where $x \in A$, $y \in A$, and $x \neq y$. There are 42 such (x, y) ordered pairs, meaning there are 21 such pairs of ordered pairs.

The probability is: ${}^{21}\!/_{C(49, 9)}$

Grading: +2 for mentioning the 49 possible pairs somewhere in their result. +2 for the choose function in the denominator. +2 for the numerator.