Computer Science Foundation Exam

August 14, 2015

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRB (Probability)	10	
3	10	PRF (Functions)	6	
4	10	PRF (Relations)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions. Justify your answers.

Each of the following questions concerns a classroom with five girls and five boys. For each part below, treat each of these 10 students as being distinguishable from one another.

(a) (3 pts) The teacher wants to create five teams, each with one girl and one boy. How many different sets of teams can the teacher create? Note: two sets of teams are different if at least one pair of students on the same team in the first set of teams is on different teams in the second set of teams.

Label the girls g_1, g_2, g_3, g_4 and g_5. Label the boys b_1, b_2, b_3, b_4 and b_5. We have 5 choices of boys to pair with g_1, followed by four remaining choices of boys to pair with g_2, etc. Thus, there are 5! sets of teams that the teacher can create. **(Grading: 3 points all or nothing.)**

(b) (5 pts) In how many ways can the class line up such that no girl is next to another girl and no boy is next to another boy?

To satisfy the criterion, we must alternate genders, but the first person in line may be either a girl or a boy. Additionally, there are 5! ways to order the girls, 5! ways to order the boys, the choices of which are independent of one another. Using the multiplication principle, our final answer is $2(5!)^2$. **(Grading: 2 pts for starting girl and starting boy, 1 pt for each 5!, 1 pt for multiplying)**

(c) (7 pts) How many subsets of students in the class contain more girls than boys? (Please give an exact numeric answer. Note that the 5^{th} row of Pascal's Triangle is 1, 5, 10, 10, 5 and 1.)

Since the number of boys and girls is equal in the class, the number of subsets containing more boys than girls is equal to the number of subsets containing more girls than boys. Let X equal the number of subsets with more girls than boys, our desired answer. Let Y equal the number of subsets with the same number of boys and girls. There are 2^{10} number of subsets in the class total. Thus, we get the equation:

$2X + Y = 2^{10} = 1024$.

We determine Y by breaking up our counting into six cases: subsets with k girls and k boys, for $0 \le k \le 5$. Since these choices are independent, we have $Y = \sum_{k=0}^{5} \binom{5}{k}^2 = 1 + 25 + 100 + 100 + 25 + 1 = 252$.

It follows that $X = \frac{1024-252}{2} = 386$.

Grading: There are <u>many ways</u> to solve this. Give partial credit as necessary. Make sure you clearly understand the student's approach to solving the problem before grading.

2) (15 pts) PRB (Probability)

(a) (7 pts) 2% of the population has disease A. Given that a patient has disease A, a test, T, correctly identifies that the patient has the disease 95% of the time. given that a patient does NOT have disease A, T correctly identifies that the patient doesn't have the disease 80% of the time. You've taken test T and have tested positive for disease A. What is the actual probability that you have the disease?

Let a be the event that a randomly selected person has disease A and let t be the event that a randomly selected person tests positive for disease A using test T. We know the following:

$$p(a) = .02, p(t|a) = .95, p(\bar{t}|\bar{a}) = .8$$

It naturally follows that, $p(\bar{t}|a) = 1 - p(t|a) = 1 - .95 = .05$ and similarly, $p(t|\bar{a}) = .2$.

We desire to find $p(a|t) = \frac{p(a \cap t)}{p(t)}$. Using a tree diagram, we can solved for both of these quantities:

$$p(a \cap t) = p(a) \times p(t|a) = .02 \times .95 = .019$$

$$p(t) = p(a)p(t|a) + p(\bar{a})p(t|\bar{a}) = .02 \times .95 + .98 \times .2 = .019 + .196 = .215$$

It follows that $p(a|t) = \frac{p(a \cap t)}{p(t)} = \frac{.019}{.215} \sim .0884 (8.84\%)$.

Grading: 2 pts for filling in opposite probabilities, 1 pt for identifying what needs to be found (includes division), 2 pt for p(t), 2 pts for p(a and t)

(b) (8 pts) Consider a partition of an array of 2n+1 distinct integers where the partition element is randomly chosen. (Note: this is where we place all integers less than the partition element on the left of it and all the integers greater than the partition element on the right of it.) Let L equal the number of values less than the partition element and R equal the number of values greater than the paritition element. (Thus, L + R = 2n.) Calculate the expected value of |L - R|.

Each of the 2n+1 integers is chosen as the partition element with probability $\frac{1}{2n+1}$. Each of the ordered pairs (L, R) associated with the partition elements are (2n, 0), (2n-1, 1), ..., (n,n), ..., (1, 2n-1), (0, 2n). The corresponding values of |L - R| are 2n, 2n-2, ..., 0, 2, 4, ..., 2n-2 and 2n. Basically, each positive even value from 2 to 2n appears on the list twice while 0 appears once. Plugging into the formula for expectation, we find:

$$\boldsymbol{Exp(|L - R|) = \frac{2\sum_{k=1}^{n} 2k}{2n+1} = \frac{4\sum_{k=1}^{n} k}{2n+1} = \frac{4n(n+1)}{2(2n+1)} = \frac{2n(n+1)}{2n+1}}.$$

Grading: 2 pts for definition of expectation, 2 pts for 1/(2n+1) probability, 4 pts for algebra.

3) (10 pts) PRF (Functions)

Let $f(x) = \sqrt{3x^2 + 5x + 7}$ and $g(x) = 2^{x^2}$, both with the domain $x \geq 0$.

(a) (7 pts) What is $g(f(x))$? (Simplify your answer for full credit.)

$$g(f(x)) = g\left(\sqrt{3x^2 + 5x + 7}\right) = 2^{\sqrt{3x^2+5x+7}^2} = 2^{3x^2+5x+7}$$

Grading: 4 pts for plugging into definition, 3 pts for sqrt/sqr simplification, other simplifications aren't necessary.

(b) (3 pts) What is the range of $g(f(x))$, given that its domain is all reals values of $x \geq 0$.

Note that $f(x)$ is a continuously increasing function and that $f(0) = \sqrt{7}$. Since $g(f(x))$ is also continuously increasing, it follows that its minimum value is $g(f(0)) = 2^7 = 128$. Thus, the range of $g(f(x))$ is [128, ∞).

Grading: 2 pts for low bound, 1 pt for high bound, no explanation necessary.

4) (10 pts) PRF (Relations)

(a) (5 pts) Let A = {1, 2, 3, 4}, B = {x, y, z} and C = {m, n}. Let R = {(1, x), (1, z), (2, y), (4, x), (4, y), (4, z)} and S = {(x, m), (y, m), (y, n)}. What is $S \circ R$?

$$S \circ R = \{(1, m), (2, m), (2, n), (4, m), (4, n)\}$$

Grading: +1 for each correctly included order pair, -1 penalty for listing an extra item not on the real list, cap grade at 0.

(b) (5 pts) Let A = {1, 2, 3}. There are 5 equivalence relations over $A \times A$. Explicitly list all five.

Here is the proof that there are 5 such relations:

All reflexive relations over the set A must contain (1, 1), (2, 2), (3, 3). Furthermore, due to symmetry, we know that (1, 2) is in the relation iff (2, 1) is, (1, 3) is in the relation iff (3, 1) is, and (2, 3) is in the relation iff (3, 2) is. Thus, we only have freedom in selecting whether or not (1, 2), (1, 3) and (2, 3) are in the relation. We must determine how many of these 2^3 = 8 possible relations are transitive. We are safe in including none, one or three of these ordered pairs. Any choice of exactly two of these will break transitivity. (For example, if we choose (1, 2) and (1, 3), that means our relation also has (3, 1). But note that it has (3, 1) and (1, 2) but doesn't contian (2, 3). Thus, out of the 8 possible relations, only three are not transitive.

It follows that there are 5 relations over A that are equivalence relations:

{(1, 1), (2, 2), (3, 3)}

{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)}

{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)}

{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)}

Grading: +1 for each correctly listed relation, -1 penalty for listing a relation that isn't an equivalence relation, cap grade at 0.