Computer Science Foundation Exam

Aug 10, 2012

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	6	
3	15	PRF (Sets)	10	
ALL	40	---	26	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Prove, using mathematical induction, that for all positive integers n, we have

$$1+4+7+10+...+(3n-2)=\frac{n(3n-1)}{2}$$

<u>Solution</u>

Base case: n = 1. The left-hand side of this equation evaluated at n = 1 is 1.
The right hand side of this equation evaluated at n = 1 is 1.
Thus, the equation holds for n = 1. (2 pts)

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$1+4+\cdots+(3k-2)=\frac{k(3k-1)}{2} \qquad \textbf{(2 pts)}$$

Inductive step: Prove for n = k + 1 that

$$1+4+\cdots+(3(k+1)-2)=\frac{(k+1)(3(k+1)-1)}{2} \textbf{ (3 pts)}$$

$$\begin{aligned}1+4+\cdots+(3(k+1)-2) &= \frac{k(3k-1)}{2}+(3k+1)\\ &= \frac{k(3k-1)+2(3k+1)}{2}\\ &= \frac{3k^2+5k+2}{2}\\ &= \frac{(3k+2)(k+1)}{2}\\ &= \frac{(k+1)(3(k+1)-1)}{2} \quad \textbf{(8 pts)}\end{aligned}$$

This completes proving the inductive step. It follows that the original assertion holds for all positive integers n.

2) (10 pts) PRF (Logic)

Show that $(p \to r) \land (q \to r)$ and $(p \lor q) \to r$ are logically equivalent using the laws of logic equivalence and the definition of the conditional statement only. Show each step and state which rule is being used.

<u>Solution</u>

$(p \to r) \land (q \to r)$		
$= (\neg p \lor r) \land (\neg q \lor r)$	**Definition of Implication**	**(1 pts)**
$= ((\neg p \lor r) \land \neg q) \lor ((\neg p \lor r) \land r)$	**Distibutive Law**	**(1 pts)**
$= ((\neg p \lor r) \land \neg q) \lor r$	**Absorption Law**	**(1 pts)**
$= (\neg p \land \neg q) \lor (r \land \neg q) \lor r$	**Distibutive Law**	**(2 pt)**
$= \neg(p \lor q) \lor (r \land \neg q) \lor r$	**DeMorgan's Law**	**(2 pts)**
$= \neg(p \lor q) \lor r$	**Absorption Law**	**(1 pts)**
$= (p \lor q) \to r$	**Definition of Implication**	**(2 pts)**

Grading: Give positive points for steps in the correct direction, roughly the steps are outlined above. Students may use more steps and rules. In this case, award points based on the relative progress of their intermediate steps as compared to what is shown above.

Here is an alternate and quicker solution:

$(p \to r) \land (q \to r)$		
$= (\neg p \lor r) \land (\neg q \lor r)$	**Definition of Implication**	**(2 pts)**
$= (\neg p \land \neg q) \lor r$	**Distributive Law**	**(3 pts)**
$= \neg(p \lor q) \lor r$	**DeMorgan's Law**	**(2 pts)**
$= (p \lor q) \to r$	**Definition of Implication**	**(3 pts)**

3) (15 pts) PRF (Sets)

Let A, B and C be arbitrary sets taken from the universe of integers. Prove the following assertion using direct proof.

Note: You may **<u>not</u>** use the "truth table" or "membership" method for the proof.

$$A \cap [[(B \cap \overline{C}) \cup C] \cap [(B \cap \overline{C}) \cup \overline{B}]] = [A \cap (B \cup C)] - (B \cap C)$$

Solution.

$A \cap [[(B \cap \overline{C}) \cup C] \cap [(B \cap \overline{C}) \cup \overline{B}]]$

$= A \cap [[(B \cup C) \cap (\overline{C} \cup C)] \cap [(B \cup \overline{B}) \cap (\overline{C} \cup \overline{B})]]$ by the distributive law **(3 pts)**

$= A \cap [[(B \cup C) \cap U] \cap [(U \cap (\overline{C} \cup \overline{B})]]$ by definition of U (the universe) **(2 pts)**

$= A \cap [(B \cup C) \cap (\overline{C} \cup \overline{B})]$ because $U \cap S = S$ for all sets S **(2 pts)**

$= [A \cap (B \cup C)] \cap (\overline{C} \cup \overline{B})$ by the associative law **(2 pts)**

$= [A \cap (B \cup C)] - \overline{(\overline{C} \cup \overline{B})}$ by the definition of set difference (and double negation) **(3 pts)**

$= [A \cap (B \cup C)] - (C \cap B)$ by DeMorgan's Laws **(2 pts)**

$= [A \cap (B \cup C)] - (B \cap C)$ by the commutative law. **(1 pt)**