Computer Science Foundation Exam

August 12, 2011

Section II B

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRF (Relations)	10	
3	15	PRF (Functions)	10	
4	15	NTH (Number Theory)	10	
ALL	60	---	40	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

(a) (9 pts) Alfred, Bonnie, Cammy, Don, Esther and Frank must stand in line. However, Alfred, Bonnie and Cammy all can't stand next to each other. (Namely, Alfred can stand next to Bonnie or Cammy, but all three can not be contiguously arranged in line.) How many different ways can the line be arranged. (Note: the line has a distinct front, so A, B, D, C, F, E is a different line ordering than E, F, C, D, B, A.)

(b) (6 pts) A store has 10 different computers and 8 different printers. You need to buy 5 computers and 5 printers for the office. How many different combinations of computers and printers can you buy? (A combination of 10 items is different than another combination of 10 items, if one item in the first combination is not included in the second combination.)

(a) The total number of ways the people can stand in line is 6! without any restrictions. (2 pts) From this number, we must subtract out each arrangement where A, B and C are contiguously arranged. (1 pt)

Let's consider each way in which A, B and C occupy contiguous positions. They can be located in positions (1,2,3), (2,3,4), (3,4,5) or (4,5,6) (2 pts). In each of these configurations, there are 3! ways to arrange A, B and C (2 pts). There are ALSO 3! ways to arrange D, E and F amongst the three remaining locations (2 pts). This means the number of forbidden arrangements is:

4 x 3! x 3! = 4 x 6 x 6 = 144.

Thus, the final answer is 6! – 4 x 3! x 3! = 720 – 144 = 576.

Adjust the grading accordingly for different solutions. Solutions may be left in factorial form.

(b) You can buy the 5 computers in $\binom{10}{5}$ ways, since you are choosing 5 computers out of 10 (2 pts). You can buy the 5 printers in $\binom{8}{5}$ ways (2 pts). For combinations of computers and printers, multiply these two, since each set of computers can be paired up with each distinct set of printers (2 pts). Thus, the final answer is $\binom{10}{5}\binom{8}{5}$ = 14,112. (Solutions can be left in combinatorial form.)

2) (15 pts) PRF (Relations)

Let the relation R, over the positive integers, be defined as follows:

R = { (a, b) | b = an, for some positive integer n }

Prove that R is a partial ordering relation.

We must prove that R is reflexive, anti-symmetric and transitive. (3 pts)

To prove R is reflexive, we must show that (a,a) ∈ R (1 pt). Since a = 1 x a, it follows that (a,a) ∈ R (2 pts). Thus, R is reflexive. (In essence, (a,a) is in R because a is a multiple of a.)

To prove that R is anti-symmetric, we must prove the following:

if (a,b) ∈ R and (b,a) ∈ R, then a = b. (1 pt)

We use direct proof. If (a,b) ∈ R, then b = an, for some positive integer n. (1 pt)
If (b,a) ∈ R, then a = bm, for some positive integer m. (1 pt)

Substitute a = bm into the first equation to yield: b = (bm)n. (1 pt) Dividing both sides of this equation by b (which is guaranteed not to be zero), yields 1 = mn. If m and n are both positive integers, it follows that both MUST be equal to 1. This infers that a = b, as desired. (1 pt)

To prove that R is transitive, we must prove the following:

if (a,b) ∈ R and (b, c) ∈ R, then (a,c) ∈ R. (1 pt)

Using direct proof, we have b = an and c = bm, for some positive integers n and m, respectively. (1 pt)

Substitute b = an into the second equation to yield: (1 pt)

c = (an)m
c = (nm)a. (1 pt)

Since n and m are integers, nm is as well. It follows that c is a multiple of a and (a, c) ∈ R, as desired.

3) (15 pts) PRF (Functions)

(a) (7 pts) Prove that the function $f(x) = x^3 - 3x^2 + 2x + 5$, where the domain is the real numbers is NOT injective.

(b) (8 pts) Let f and g be functions such that f: A → B and g: B → C, where f is injective and g is surjective. Prove that the composition function, g◦f, may NOT be surjective.

(a) f(0) = 5 and f(1) = $1^3 - 3(1^2) + 2(1) + 5 = 1 - 3 + 2 + 5 = 5$. Thus, f is not injective, since f(0) = f(1). (Grading – 3 pts for correctly calculating f(a) and f(b), for some values a and b. 1 point for stating that these two things are equal so the function isn't injective.)

(b) Let A = {1}, B = {2, 3} and C = {4,5}. Let f = {(1,2)} and g = {(2,4), (3,5)}. In this case, it g◦f = {(1,4)}, which is NOT a surjective function, since there exists no a such that g◦f(a) = 5. Thus, we have found an example where f is injective, g is surjective, but their composition, g◦f, is NOT surjective.

Grading: 3 pts for listing sets A, B and C. 1 pt for listing a function f. 1 pt for listing a function g. 1 pt for correctly calculating g◦f, 2 pts if g◦f isn't surjective.

4) (15 pts) NTH (Number Theory)

(a) (10 pts) Prove that $\sqrt[3]{7}$ is an irrational number. You may utilize the following lemma: if $p \mid x^n$, where x and n are positive integers and p is a prime number, then $p \mid x$.

(b) (5 pts) Determine the least common multiple of 72 and 96.

(a) Assume the opposite that $\sqrt[3]{7}$ is rational (1 pt). Then we can express the quantity as a fraction in lowest terms. Let this be $\frac{p}{q}$, where gcd(p, q) = 1 (1 pt).

$\sqrt[3]{7} = \frac{p}{q}$, now, cube both sides.
$7 = \frac{p^3}{q^3}$, multiply both sides by q^3. (1 pt)
$7q^3 = p^3$, given the lemma since $7 \mid p^3$, it follows that $7 \mid p$. Let p = 7a. (2 pts)
$7q^3 = (7a)^3$, multiply this out (1 pt)
$7q^3 = 343a^3$, divide both sides by 7. (1 pt)
$q^3 = 49a^3$, given the lemma since $7 \mid q^3$, it follows that $7 \mid q$. Let q = 7b. (2 pts)

This leads to a contradiction because 7 | p and 7 | q, which contradicts the assumption that gcd(p, q) = 1 (1 pt). Thus, our initial assumption, that we can express $\sqrt[3]{7}$ as a fraction in lowest terms must be erroneous. It follows that $\sqrt[3]{7}$ is irrational.

(b) 96 = 1 x 72 + 24
72 = 3 x 24, thus, gcd(72, 96) = 24.

$$lcm(72, 96) = \frac{72 \times 96}{gcd(72,96)} = \frac{72 \times 96}{24} = 3 \times 96 = 288.$$

Grading: gcd – 2 pts, lcm = ab/gcd – 2 pts, actual numerical value – 1 pt