Computer Science Foundation Exam

August 14, 2009

Section II B

DISCRETE STRUCTURES SOLUTIONS

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Name: __________________________________

PID: __________________________________

In this section of the exam, there are four (4) problems.
You must do ALL of them.
Each counts for 15% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

4) (CTG) Counting (15 pts)

(a) (5 pts) How many permutations are there of the letters in the word ENGINEERING?

(b) (10 pts) An ice cream parlor sells six flavors of ice cream: vanilla, chocolate, strawberry, cookies and cream, mint chocolate chip, and chocolate chip cookie dough. How many combinations of fewer than 20 scoops are there? (Note: two combinations count as distinct if they differ in the number of scoops of at least one flavor of ice cream.)

(a) Using the permutation formula, we get $\frac{11!}{3!3!2!2!1!}$, since there are 3 Es, 3 Ns, 2 Gs, 2 Is and 1 R in ENGINEERING. (Note: The answer is identical if we leave off the 1!, since this is equal to 1.) Grading – 2 pts for the numerator, 1 pt for a fraction, and 2 pts for the denominator.

(b) This is a slight variation on the standard combinations with repetition problem. The difference here is that we are not trying to buy exactly 19 scoops of ice cream, but 19 or fewer scoops. We can solve this problem by introducing a 7th flavor, called "no-flavor" ice cream. Now, imagine trying to buy exactly 19 scoops of ice cream from the 7 possible flavors (the six listed an "no-flavor"). Any combination with only 10 real scoops would be assigned 9 "no-flavor" scoops, for example. There is a one-to-one correspondence between each possible combination with 19 or fewer scoops from 6 flavors as there are to 19 "scoops" from 7 flavors. Thus, using the formula for combination with repetition with 19 items from 7 types, we find the number of ways to buy the scoops is $\binom{19+7-1}{19} = \binom{25}{19} = \binom{25}{6}$. (Grading – 4 pts for mentioning the idea of an extra flavor, 4 pts for attempting to apply the correct formula, 2 pts for getting the correct answer. If a sum is given instead of a closed form, give 6 points out of 10.)

5) (PRF) Relations (15 pts)

Define the following relation R on Z^+: $R = \{(x, y) \mid \exists c \in Z^+, y = cx\}$.

Show that R is a partial-ordering relation.

We must show that R is reflexive, anti-symmetric and transitive. (1 pt)

To show that R is reflexive, consider an arbitrary ordered pair (a,a). Since a = 1(a), it follows that (a,a) ∈ R. (2 pts)

To show that R is anti-symmetric, consider an ordered pair (a,b)∈ R, where (b,a)∈ R. This means that there exist positive integers c and d such that

b = ca (1 pts)
a = db (1 pts)

Now, substitute for a in the first equation to yield b = c(db). Now, divide by b, (which is valid since b is positive) to yield 1 = cd. Since c and d are both positive integers, it follows that c = d = 1. Substituting, this proves that a = b as desired. (4 pts)

To show that R is transitive, we must show that if (a,b) ∈ R and (b,c) ∈ R, then (a,c) ∈ R.

Using the given information, for some positive integers d and e, we have:

b = da (1 pt)
c = eb (1 pt)

Substituting for b in the second equation we get: c = e(da), so c = (ed)a. Since e and d are positive integers, so is ed. Thus, it follows that (a,c) ∈ R, since c is a positive multiple of a. (4 pts)

6) (PRF) Functions (15 pts)

Let *f* and *g* be functions such that *f*: A→B and *g*: B→C. Prove or disprove the following:

(a) (8 pts) If $g \circ f$ is surjective, then *g* is surjective.

(b) (7 pts) If *g* is surjective, then $g \circ f$ is surjective.

(a) We can prove this through contradiction.

Assume the opposite, that g is NOT surjective. (1 pt)

Then there exists some element z ∈ C such that there is NO element y∈ B such that g(y) = z. (2 pts)

Now, since $g \circ f$ is surjective, this means that there is some element x such that g(f(x)) = z. (2 pts)

Because f's co-domain is B, it follows that f(x) ∈ B. (1 pt)

But, this contradicts the fact that there is no element y in B such that g(y) = z. Thus, it follows that the initial assume is incorrect and g must be surjective. (2 pts)

Grading: Note, there are other ways to prove this, correspondingly award points.

(b) This is false. (2 pts) Consider the following counter-example:

A = {1} B = {2, 3} C = {4, 5}. (2 pts)
Let the function f = {(1,2)} and the function g = {(2,4), (3,5)}. (2 pts)

In this example, g is surjective, since both 4 and 5 are covered. But, $g \circ f$ = {(1,4)} and is NOT surjective, since it does NOT cover 5. (1 pt)

Grading: There are many, many examples that work. Give 2 points for stating the result. 4 points for fully writing out the example and 1 point for verifying that it is a counter-example.

7) (NTH) Number Theory (15 pts)

(a) (5 pts) Write 11! in its prime factorized form.

(b) (10 pts) Find all integers n greater than 3 for which $n^2 - 9$ is a prime number. Prove that no other answers exist.

(a) 11! = 2x3x4x5x6x7x8x9x10x11 = 2x3x2^2x5x2x3x7x2^3x3^2x2x5x11 = $2^8 3^4 5^2 7^1 11^1$. (1 pt for each term in the prime factorization)

(b) $n^2 - 9 = (n - 3)(n + 3)$. For all n > 4, both factors exceed 1, proving that for all n > 4, the given expression is NEVER prime. Thus, we must just check n = 4. Indeed $4^2 - 9 = 7$, which is prime. Thus the only integer n in the given range for which $n^2 - 9$ is prime is n = 4.

Grading: 4 points for the factorization, 2 points for showing that n=4 is a solution, 4 points for arguing why no solution exists with n > 4.